LEARNING OPTIMAL SPATIALLY-DEPENDENT REGULARIZATION PARAMETERS IN TOTAL VARIATION IMAGE RESTORATION*

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ABSTRACT. We consider a bilevel optimization approach in function space for the choice of spatially dependent regularization parameters in TV image restoration models. First- and second-order optimality conditions for the bilevel problem are studied, when the spatially-dependent parameter belongs to the Sobolev space $H^1(\Omega)$. A combined Schwarz domain decomposition-semismooth Newton method is proposed for the solution of the full optimality system and local superlinear convergence of the semismooth Newton method is analyzed. Exhaustive numerical computations are finally carried out to show the suitability of the approach.

1. INTRODUCTION

Let $\Omega \in \mathbb{R}^2$ be an open, bounded domain and $f \in L^p(\Omega)$, for some $p > 2$, a given noisy image. For removing the noise, Total Variation (TV) regularization is frequently considered (see e.g., [1, 3–5]). The idea is to reconstruct a denoised version u of f by minimizing the generic functional

$$\mathcal{F}(u) = |Du|(\Omega) + \int_\Omega \lambda\, \phi(u, f)dx$$

where $|Du|(\Omega) = \sup_{v \in C_0^\infty(\Omega,\mathbb{R}^2), \|v\| \le 1} \int_\Omega u \nabla \cdot v\, dx$ is the total variation of u in Ω, λ is a positive parameter and ϕ is a suitable fidelity function, dependent on the type of noise included in f.

The parameter λ can be either a positive constant or a spatially dependent function $\lambda : \Omega \to \mathbb{R}^+$. If $\lambda \in \mathbb{R}^+$, the parameter serves as a weight between the fidelity measure and the TV-regularizing term. On the other hand, if λ is considered as spatially dependent, i.e., $\lambda : \Omega \to \mathbb{R}^+$, it can also reflect information on possibly non-homogeneous noise in the image, as well as making a difference between regularization of small and large scale features in the image. Hence, λ has a key role in spatially balancing the amount of regularization. Spatially dependent parameters have been considered in the recent papers [2, 8, 19, 20].

The choice of an appropriate regularization parameter λ is a difficult task and has been the subject of many research efforts (see, e.g., [8, 10–12, 14, 26, 27, 29]). In [7], a bilevel optimization approach in function space was proposed for learning the weights between noise model and TV-regularization. In the flavour of supervised machine learning, the approach presupposes the existence of a training set of clean and noisy images. Existence of Lagrange multipliers was proved and an optimality system characterizing the solution was obtained. The analytical results hold both for $\lambda \in \mathbb{R}^+$ and $\lambda : \Omega \to \mathbb{R}^+$, while a solution algorithm was only designed for solving the bilevel optimization problem with $\lambda \in \mathbb{R}^+$. A related approach for finite-dimensional variational problems was proposed in [18].

In this article we consider a bilevel optimization approach with a spatially dependent parameter $\lambda \in H^1(\Omega)$, and investigate first- and second-order optimality conditions for the bilevel problem. In addition to the nonsmooth lower level denoising problems, a positivity

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constraint on the functional parameter ($\lambda \geq 0$ a.e. in Ω) has to be imposed to guarantee well-posedness. These elements lead to a nonlinear and nonsmooth first-order optimality system with complementarity relations. For proving second order sufficient optimality conditions, we improve previous Gâteaux differentiability results [7] and show that the solution mapping is actually twice Fréchet differentiable. We then define a suitable cone of critical directions and utilize a contradiction argument.

Since the resulting optimality system involves several coupled PDEs (twice the size of the training set), the efficient numerical solution of the problem becomes challenging. We consider a combined Schwarz domain decomposition-semismooth Newton approach, where the domain Ω is subdivided into overlapping subdomains Ω_i with "optimized" transmission conditions (see, e.g., [13,24,25]). In the spirit of [9], we apply Schwarz domain decomposition methods directly to the nonlinear optimality system rather than to a linearisation of it, and solve, in each subdomain, a reduced nonlinear and nonsmooth optimality system. We propose a semismooth Newton algorithm for the solution of each subdomain system and analyze the local superlinear convergence of the method.

The outline of the paper is as follows. In Section 2 the bilevel optimization problem is stated and analyzed. The analysis involves differentiability properties of the solution operator and the derivation of first and second order optimality conditions. The numerical treatment of the problem is considered in Section 3. The discretization of the problem is described and the domain decomposition and semismooth Newton algorithms are presented. Also the convergence analysis of the semismooth Newton method is carried out. Finally, in Section 4 an exhaustive numerical experimentation is presented. We compare our approach with other spatially-dependent approaches and apply it to problems with large training sets.

2. The bilevel optimization problem in function space

Given a training set $(u_i^\dagger, f_i)$, $i = 1, \dots, N$, of truth and noisy images, respectively, the bilevel optimization problem under consideration reads as follows: Find a minimizer $(u_1^*, \dots, u_N^*, \lambda^*) \in [H_0^1(\Omega)]^N \times H^1(\Omega)$ of the problem

$$\min_{(u_1,\dots,u_N,\lambda)\in[H_0^1(\Omega)]^N\times H^1(\Omega)} J(u_1,\dots,u_N,\lambda) := \sum_{i=1}^N \|u_i - u_i^\dagger\|_{L^2}^2 + \beta\|\lambda\|_{H^1(\Omega)}^2 \tag{2.1a}$$

subject to:

$$\langle e_i(u_i,\lambda), v\rangle_{H^{-1},H_0^1} = \mu\big(Du_i, Dv\big)_{L^2} + \big(h_\gamma(Du_i), Dv\big)_{L^2} + \int_\Omega \lambda\phi'(u_i,f_i)v\,dx = 0 \quad \text{for all } v \in H_0^1(\Omega),\ i = 1,\dots,N, \tag{2.1b}$$

$$\lambda \geq 0 \quad \text{a.e. in } \Omega, \tag{2.1c}$$

where N is the size of the training set of images, $0 < \mu \ll 1$, $e_i : H_0^1(\Omega) \times H^1(\Omega) \to H^{-1}(\Omega)$, for $i = 1, \dots, N$ and

$$\phi(u_i, f_i) = \|u_i - f_i\|_{L^2}^2,\ i = 1,\dots,N.$$

In order to simplify the presentation, we focus hereafter on the case $N = 1$. The results are, however, still valid for larger training sets, as will be commented in Section 4.

The Huber C^2-regularizing function h_γ is given by:

$$h_\gamma(z) = \begin{cases} \frac{z}{|z|} & \text{if } \gamma|z| \geq b \\ \frac{z}{|z|}\Big\{\frac{2\gamma-1}{4\gamma} + \frac{\gamma|z|}{2} - \frac{\gamma}{2}\big(\gamma|z|-a\big)\big(\gamma|z|-b\big) + \frac{\gamma^3}{2}\big(\gamma|z|-a\big)^2\big(\gamma|z|-b\big)^2\Big\} & \text{if } a < \gamma|z| \leq b \\ \gamma z & \text{else,} \end{cases} \tag{2.2}$$

where $a = 1 - \frac{1}{2\gamma}$ and $b = 1 + \frac{1}{2\gamma}$. This function locally regularizes the subgradient of the TV-norm. The next result involves some properties of h_γ, which will be used along the paper.

Lemma 2.1. *There exist constants $L_\gamma, C_\gamma, M_\gamma > 0$ only dependent in γ such that*

a) *For all $z, \hat{z}, \xi, \tau \in \mathbb{R}^N \times \mathbb{R}^N$ ($N \in \mathbb{N}^*$),*

$$|R(z,\hat{z},\xi,\tau)| := \left| h''_\gamma(z)[\xi,\tau] - h''_\gamma(\hat{z})[\xi,\tau] \right| \le L_\gamma |z - \hat{z}||\xi||\tau|. \tag{2.3}$$

b) *For all $u, w \in \mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ we have $h'_\gamma(u)[w] \in \mathbb{L}^2(\Omega)$ and*

$$\big([h'_\gamma(u) - h'_\gamma(\hat{u})]w, v\big)_{L^2} \le C_\gamma \|u - \hat{u}\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2} \|v\|_{\mathbb{L}^2}, \quad \forall \hat{u}, v \in \mathbb{L}^2(\Omega).$$

c) *For all $u \in \mathbb{L}^2(\Omega)$ we have $h''_\gamma(u) \in \mathbb{L}^\infty(\Omega) := \big(L^\infty(\Omega)\big)^{2\times 2}$. Moreover,*

$$\|h''_\gamma(u) - h''_\gamma(\hat{u})\|_{\mathbb{L}^\infty(\Omega)} \le M_\gamma \|u - \hat{u}\|_{\mathbb{L}^2}.$$

Proof. The proof is included in the Appendix. □

From [7] we know that for each fixed $\gamma > 0$, there exists an optimal solution for problem (2.1). Denoting by $G : H^1(\Omega) \to H^1_0(\Omega)$ the solution operator $G(\lambda) = u$, where u is solution of equation (2.1b) corresponding to $\lambda \in H^1(\Omega)$, it has been shown in [7] that the operator is Gâteaux differentiable. In the next theorem we improve that result and prove that the solution operator is actually twice Fréchet differentiable under some (nonrestrictive) data regularity assumptions.

Theorem 2.1. *Let $f \in L^p(\Omega)$, for some $p > 2$, and $V_{ad} := \{v \in H^1(\Omega) : v \ge 0 \quad a.e. \quad \Omega\}$. The* solution *operator $G : V_{ad} \to H^1_0(\Omega)$, $G : \lambda \mapsto u(\lambda)$, where $u(\lambda)$ is the solution to* (2.1*b*), *is Fréchet differentiable and its derivative at $\lambda \in V_{ad}$, in any direction $\xi \in H^1(\Omega)$, is given by $z^\xi_\lambda = G'(\lambda)\xi \in H^1_0(\Omega)$, which corresponds to the unique solution of the linearized equation:*

$$\mu\big(Dz, Dv\big)_{L^2} + \big(h'_\gamma(Du)^* Dz, Dv\big)_{L^2} + 2\int_\Omega \lambda z v + 2\int_\Omega \xi(u - f)v = 0, \forall v \in H^1_0(\Omega). \tag{2.4}$$

Furthermore, if $f \in L^\infty(\Omega)$ and $u(\lambda) \in C^{1,\beta}(\Omega)$, for some $\beta \in (0,1)$, then G is twice Fréchet differentiable and its second derivative is given by $w^{(\xi,\zeta)}_\lambda \in H^1_0(\Omega)$ solution of

$$\begin{aligned} &\mu\big(Dw^{(\xi,\zeta)}_\lambda, Dv\big) + \big(h'_\gamma(Du) Dw^{(\xi,\zeta)}_\lambda, Dv\big) + 2\int_\Omega \lambda w^{(\xi,\zeta)}_\lambda v dx \\ &\quad + \big(h''_\gamma(Du)^*[Dz^\xi_\lambda, Dz^\zeta_\lambda], Dv\big) + 2\int_\Omega \zeta z^\xi_\lambda v dx + 2\int_\Omega \xi z^\zeta_\lambda v dx = 0, \quad \forall v \in H^1_0(\Omega). \end{aligned} \tag{2.5}$$

Proof. For arbitrary but fixed $\lambda, \xi \in V_{\mathrm{ad}}$, we denote by u and u_ξ the corresponding solutions to (2.1b) with λ and $\lambda + \xi$, respectively. By monotonicity techniques (see [7]), we obtain the existence and uniqueness of a solution $z^\xi_\lambda \in H^1_0(\Omega)$ to (2.4) and we also have that

$$\|u_\xi - u\|_{H^1_0} = O(\|\xi\|_{H^1}), \quad \|z^\xi_\lambda\|_{H^1_0} = O(\|\xi\|_{H^1}). \tag{2.6}$$

By taking the difference between (2.1b), with λ and $\lambda + \xi$, and (2.4) we get that

$$\begin{aligned} &\mu\big(D(u_\xi - u - z^\xi_\lambda), Dv\big)_{L^2} + \big(h_\gamma(Du_\xi) - h_\gamma(Du) - h'_\gamma(Du) Dz^\xi_\lambda, Dv\big)_{L^2} \\ &\qquad + 2\int_\Omega \lambda(u_\xi - u - z^\xi_\lambda)v + 2\int_\Omega \xi(u_\xi - u)v = 0, \quad \forall v \in H^1_0(\Omega). \end{aligned}$$

Introducing $\eta := u_\xi - u - z^\xi_\lambda$, we can write the last equation as follows

$$\begin{aligned} \mu\big(D\eta, Dv\big)_{L^2} + \big(h'_\gamma(Du) D\eta, Dv\big)_{L^2} + 2\int_\Omega \lambda \eta v &= -2\int_\Omega \xi(u_\xi - u)v \\ &\quad - \big(h_\gamma(Du_\xi) - h_\gamma(Du) - h'_\gamma(Du) D(u_\xi - u), Dv\big)_{L^2}, \quad \forall v \in H^1_0(\Omega). \end{aligned}$$

Taking $v = \eta$ and using the monotonicity of $h'_\gamma(Du)$ and $\lambda \geq 0$ a.e. Ω, one gets, for some constant $C_3 > 0$, that

$$\|\eta\|^2_{H^1_0} \leq \left| \big(h_\gamma(Du_\xi) - h_\gamma(Du) - h'_\gamma(Du)D(u_\xi - u), D\eta\big)_{L^2} \right| + C_3 \|\xi\|_{H^1} \|u_\xi - u\|_{H^1_0} \|\eta\|_{H^1_0}.$$

Due to the twice-differentiability of h_γ, we obtain

$$|\eta\|_{H^1_0} \leq C_4 \left(\|u_\xi - u\|^2_{W^{1,p}} + \|\xi\|_{H^1} \|u_\xi - u\|_{H^1_0} \right), \tag{2.7}$$

for all $p > 2$ and some constant $C_4 > 0$. Thanks to [15, Thm. 1], there is some $q > 2$ such that

$$\|u_\xi - u\|_{W^{1,q}} = O(\|\xi\|_{H^1}).$$

From the latter and estimates (2.6), it then follows that $\|\eta\|_{H^1_0} = O(\|\xi\|^2_{H^1})$. The last relation ensures the Fréchet differentiability of G and that $z^\xi_\lambda = G'(\lambda)\xi$.

Concerning the second derivative, for $\xi, \zeta \in H^1(\Omega)$ and $\lambda \in V_{\text{ad}}$, let $z^\xi_\lambda = G'(\lambda)\xi$ and $z^\zeta_\lambda = G'(\lambda)\zeta$ be the corresponding linearized solutions. We denote by $w^{(\xi,\zeta)}_\lambda$ the solution of the following equation:

$$\begin{aligned} \mu\big(Dw, Dv\big) + \big(h'_\gamma(Du)Dw, Dv\big) + 2\int_\Omega \lambda w v \\ + \big(h''_\gamma(Du)^*[Dz^\xi_\lambda, Dz^\zeta_\lambda], Dv\big) + 2\int_\Omega \zeta z^\xi_\lambda v dx + 2\int_\Omega \xi z^\zeta_\lambda v = 0, \forall v \in H^1_0(\Omega). \end{aligned} \tag{2.8}$$

Existence and uniqueness of $w^{(\xi,\zeta)}_\lambda$, for every given $\xi, \zeta \in V$ and $\lambda \in V_{\text{ad}}$, follows from the Lax-Milgram theorem.

Let $\zeta \in H^1(\Omega)$ be such that $\lambda_\zeta := \lambda + \zeta \in V_{\text{ad}}$, and let $z^\xi_{\lambda_\zeta} := G'(\lambda_\zeta)\xi$ and u_ζ be the solution to (2.1b) corresponding to λ_ζ. Taking the difference between (2.4) for z^ξ_λ and $z^\xi_{\lambda_\zeta}$, one gets

$$\begin{aligned} \mu\big(D(z^\xi_{\lambda_\zeta} - z^\xi_\lambda), Dv\big) + \big(h'_\gamma(Du)D(z^\xi_{\lambda_\zeta} - z^\xi_\lambda), Dv\big) + 2\int_\Omega \lambda(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)v \\ + \big(\big[h'_\gamma(Du_\zeta) - h'_\gamma(Du)\big]Dz^\xi_{\lambda_\zeta}, Dv\big) + 2\int_\Omega \zeta z^\xi_{\lambda_\zeta} v + 2\int_\Omega \xi(u_\zeta - u)v = 0, \forall v \in H^1_0(\Omega). \end{aligned} \tag{2.9}$$

Testing (2.9) with $v = z^\xi_{\lambda_\zeta} - z^\xi_\lambda$, we have

$$\begin{aligned} \|z^\xi_{\lambda_\zeta} - z^\xi_\lambda\|^2_{H^1_0} \leq C_5 \Bigg\{ \left| \big(\big[h'_\gamma(Du_\zeta) - h'_\gamma(Du)\big]Dz^\xi_\lambda, D(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)\big) \right| \\ + \left| \int_\Omega \zeta z^\xi_\lambda (z^\xi_{\lambda_\zeta} - z^\xi_\lambda) \right| + \left| \int_\Omega \xi(u_\zeta - u)(z^\xi_{\lambda_\zeta} - z^\xi_\lambda) \right| \Bigg\}, \end{aligned} \tag{2.10}$$

where $C_5 > 0$ is a constant. From Lemma 2.1, the last relation yields

$$\|z^\xi_{\lambda_\zeta} - z^\xi_\lambda\|_{H^1_0} \leq C_6 \left(\big\|u_\zeta - u\big\|_{H^1_0} \|z^\xi_\lambda\|_{H^1_0} + \|\zeta\|_V \|z^\xi_\lambda\|_{H^1_0} + \|\ \xi\|_V \|u_\zeta - u\|_{H^1_0} \right),$$

where $C_6 > 0$. Considering (2.6), the following estimate holds for some constant $C_7 > 0$

$$\|z^\xi_{\lambda_\zeta} - z^\xi_\lambda\|_{H^1_0} \leq C_7 \|\zeta\|_{H^1} \|\xi\|_{H^1}. \tag{2.11}$$

Moreover, thanks to elliptic regularity theory, it follows by a bootstrapping argument, that

$$\|z^\xi_{\lambda_\zeta} - z^\xi_\lambda\|_{W^{1,p}} \leq C_p \|\zeta\|_{H^1} \|\xi\|_{H^1}, \tag{2.12}$$

for any $p > 0$.

By setting $\tau = z^\xi_{\lambda_\zeta} - z^\xi_\lambda - w^{(\xi,\zeta)}_\lambda$ and subtracting (2.8) from (2.9), we get that

$$\mu\big(D\tau, Dv\big) + \big(h'_\gamma(Du)D\tau, Dv\big) + 2\int_\Omega \lambda\tau v =$$
$$-\big(\big[h'_\gamma(Du_\zeta) - h'_\gamma(Du)\big]D(z^\xi_{\lambda_\zeta} - z^\xi_\lambda), Dv\big) - 2\int_\Omega \zeta(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)v - 2\int_\Omega \xi(u_\zeta - u - z^\zeta_\lambda)v$$
$$-\Big(h'_\gamma(Du_\zeta)Dz^\xi_\lambda - h'_\gamma(Du)Dz^\xi_\lambda - h''_\gamma(Du)^*[Dz^\xi_\lambda, Dz^\zeta_\lambda], Dv\Big), \quad \forall v \in H^1_0(\Omega).$$

Testing with $v = \tau$ and using the ellipticity of the terms on the left hand side, we obtain that

$$\|\tau\|_{H^1_0} \le C_8\Big\{\Big\|\big[h'_\gamma(Du_\zeta) - h'_\gamma(Du)\big]D(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)\Big\|_{L^2} + \Big\|\zeta(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)\Big\|_{L^2} +$$
$$+\Big\|\xi(u_\zeta - u - z^\zeta_\lambda)\Big\|_{L^2} + \Big\|h'_\gamma(Du_\zeta)Dz^\xi_\lambda - h'_\gamma(Du)Dz^\xi_\lambda - h''_\gamma(Du)^*[Dz^\xi_\lambda, Dz^\zeta_\lambda]\Big\|_{L^2}\Big\}, \tag{2.13}$$

for some constant $C_8 > 0$.

For the first term on the left hand side, thanks to the Lipschitz continuity of h'_γ and estimate (2.12), we get that

$$\begin{aligned}\Big\|\big[h'_\gamma(Du_\zeta) - h'_\gamma(Du)\big]D(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)\Big\|_{L^2} &\le \|h'_\gamma(Du_\zeta) - h'_\gamma(Du)\|_{L^p}\Big\|z^\xi_{\lambda_\zeta} - z^\xi_\lambda\Big\|_{W^{1,q}}\\ &\le L\|u_\zeta - u\|_{W^{1,p}}\Big\|z^\xi_{\lambda_\zeta} - z^\xi_\lambda\Big\|_{W^{1,q}}\\ &\le C_9\|\zeta\|^2_{H^1}\|\xi\|_{H^1}.\end{aligned}$$

Since the solution operator has been proved to be Fréchet differentiable, it follows that $\|u_\zeta - u - z^\zeta_\lambda\|_{H^1_0} = O(\|\zeta\|^2_{H^1})$ and, thus,

$$\Big\|\xi(u_\zeta - u - z^\zeta_\lambda)\Big\|_{L^2} \le C_{10}\|\xi\|_{H^1}\|\zeta\|^2_{H^1},$$

where $C_{10} > 0$. From (2.11) it also follows that

$$\Big\|\zeta(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)\Big\| \le C_{11}\|\zeta\|^2_{H^1}\|\xi\|_{H^1},$$

for some constant $C_{11} > 0$.

For the last term on the right hand side of (2.13), we obtain that

$$\begin{aligned}\Big\|\Big(h'_\gamma(Du_\zeta) - h'_\gamma(Du) - h''_\gamma(Du)^*Dz^\zeta_\lambda\Big)Dz^\xi_\lambda\Big\|_{L^2} &\le \Big\|h''_\gamma(Du)^*D(u_\zeta - u - z^\zeta_\lambda)\Big\|_{L^r}\|Dz^\xi_\lambda\|_{L^s}\\ &+ \big\|h'_\gamma(Du_\zeta) - h'_\gamma(Du) - h''_\gamma(Du)^*D(u_\zeta - u)\big\|_{L^r}\|Dz^\xi_\lambda\|_{L^s},\end{aligned}$$

where $1/r + 1/s = 1$. Taking into account estimates (2.6) and (2.7) we get that

$$\Big\|\Big(h'_\gamma(Du_\zeta) - h'_\gamma(Du) - h''_\gamma(Du)^*Dz^\zeta_\lambda\Big)Dz^\xi_\lambda\Big\|_{L^2} \le C\|\xi\|_{H^1}\left(o(\|u_\zeta - u\|_{W^{1,p}}) + o(\|\zeta\|_{H^1})\right),$$

with $p > r$.

Now taking the estimates of $\left|\int_\Omega \zeta(z^\xi_{\lambda_\zeta} - z^\xi_\lambda)\tau\right|$, $\left|\int_\Omega \xi(u_\zeta - u - z^\zeta_\lambda)\tau\right|$ and the results in Lemma 2.1 into account (2.13), noting that $\|\eta_\zeta\|_{H^1_0} \le m_3\|\zeta\|^2_{H^1}$, we have

$$\|\tau\|_{H^1_0} \le C_{12}\|\zeta\|^2_{H^1}\|\xi\|_{H^1}$$

where $C_{12} > 0$ is a constant. The last relation ensures the twice differentiability of G and we also have that $w^{(\xi,\zeta)}_\lambda = G''(\lambda)[\xi, \zeta]$. □

Remark 2.1. *The Hölder continuity assumption on the gradient of $u(\lambda)$ is not restrictive. In fact, it may be proved under some domain and data regularity (see, e.g., [6, Thm. 2.2]).*

Based on the differentiability properties of the solution operator, a first order optimality system characterizing the optimal weight function is derived next.

Theorem 2.2. *Let $(u,\lambda) \in H_0^1(\Omega) \times V_{ad}$ be an optimal solution for (2.1). Then there exist $p \in H_0^1(\Omega)$ and $\vartheta \in L^2(\Omega)$ such that the following optimality system holds (in weak sense):*

$$-\mu\Delta u - \text{Div } q + 2\lambda(u-f) = 0 \quad \textit{in } \Omega, \tag{2.14a}$$

$$u = 0 \quad \textit{on } \Gamma, \tag{2.14b}$$

$$q = h_\gamma(Du) \quad \textit{a.e. in } \Omega, \tag{2.14c}$$

$$-\mu\Delta p + \text{Div } z + 2(\lambda p + u - u^\dagger) = 0 \quad \textit{in } \Omega, \tag{2.14d}$$

$$p = 0 \quad \textit{on } \Gamma, \tag{2.14e}$$

$$z = h'_\gamma(Du)^* Dp \quad \textit{a.e. in } \Omega, \tag{2.14f}$$

$$\vartheta = -\beta\Delta\lambda + \beta\lambda + (u-f)p \quad \textit{in } \Omega, \tag{2.14g}$$

$$\frac{\partial\lambda}{\partial\vec{n}} = 0 \quad \textit{on } \Gamma, \tag{2.14h}$$

$$\lambda \geq 0, \quad \vartheta \geq 0, \quad \vartheta\,\lambda = 0 \quad \textit{a.e. in } \Omega. \tag{2.14i}$$

Proof. Since the solution operator is differentiable, it follows, using the reduced cost functional

$$\mathcal{J}(\lambda) = \|u(\lambda) - u^\dagger\|_{L^2}^2 + \frac{\beta}{2}\|\lambda\|_{H^1(\Omega)}^2, \tag{2.15}$$

that

$$\mathcal{J}'(\lambda)(\xi-\lambda) = 2(u(\lambda) - u^\dagger, u'(\lambda)(\xi-\lambda)) + \beta(\lambda, \xi-\lambda)_{H^1} \geq 0, \qquad \forall \xi \in V_{\text{ad}}. \tag{2.16}$$

Introducing $p \in H_0^1(\Omega)$ as the unique weak solution of the adjoint equations (2.14d)-(2.14f) and replacing in (2.15), we get that

$$\beta(\lambda, \xi-\lambda)_{H^1} + \int_\Omega p(u-f)(\xi-\lambda) \geq 0, \forall \xi \in V_{\text{ad}}. \tag{2.17}$$

Inequality (2.17) corresponds to an obstacle type problem with unilateral bounds. Thanks to regularity results for obstacle problems (see [28, Thm.5.2, p.294]), it follows that $\lambda \in H^2(\Omega)$ (if $f \in L^p(\Omega)$ for some $p > 2$) and, therefore, we may define

$$\vartheta = -\beta\Delta\lambda + \beta\lambda + (u-f)p \in L^2(\Omega).$$

Integrating by parts in (2.17) we then obtain that $\left(\vartheta, \xi-\lambda\right)_{L^2} \geq 0$. From the latter and the sign of λ, we obtain that

$$\lambda \geq 0, \quad \vartheta \geq 0, \quad \vartheta\,\lambda = 0 \quad \textit{a.e.} \quad \Omega. \tag{2.18}$$

□

The complementarity condition (2.18) can also be reformulated as the following equation:

$$\vartheta = \max(0, \vartheta - \alpha\lambda), \text{ for any } \alpha > 0,$$

where max is interpreted in an almost everywhere sense. By choosing $\alpha = \beta$ and replacing in (2.14g) one gets

$$-\beta\Delta\lambda + \beta\lambda + (u-f)p - \max(0, -\beta\Delta\lambda + (u-f)p) = 0. \tag{2.19}$$

Altogether, we obtain the following system for $\mathbf{y} = (u, q, p, z, \lambda)$

$$F(\mathbf{y}) = \begin{pmatrix} -\mu\Delta u - \text{Div } q + 2\lambda(u-f) \\ u|_\Gamma \\ h_\gamma(Du) - q \\ -\mu\Delta p - \text{Div } z + 2\lambda p + 2(u-u^\dagger) \\ p|_\Gamma \\ h'_\gamma(Du)^* Dp - z \\ -\beta\Delta\lambda + \beta\lambda + (u-f)p - \max\left(0, -\beta\Delta\lambda + (u-f)p\right) \\ \partial_{\vec{n}}\lambda|_\Gamma = 0 \end{pmatrix} = 0, \tag{2.20}$$

where $F : V \to W$ with $V := H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ and $W := H^{-1}(\Omega) \times H^{1/2}(\Gamma) \times L^2(\Omega) \times H^{-1}(\Omega) \times H^{1/2}(\Gamma) \times L^2(\Omega) \times H^1(\Omega)' \times H^{-1/2}(\Gamma)$.

2.1. **Second order sufficient optimality condition.** To state a second order condition, let us start by computing the second derivatives of $J(u, \lambda)$ and the state equation operator $e(u, \lambda)$ defined in (2.1b). For $(u, \lambda) \in (\Omega) \times H^1(\Omega)$ and for all $w, \eta \in H_0^1(\Omega), l \in H^1(\Omega)$, we have:

$$J_{uu}(u,\lambda)[w]^2 = 2\|w\|_{L^2}^2, \qquad J_{\lambda\lambda}(u,\lambda)[l]^2 = 2\beta\|l\|_{H^1}^2 \tag{2.21a}$$

$$J_{u\lambda}(u,\lambda) = 0, \qquad J_{\lambda u}(u,\lambda) = 0 \tag{2.21b}$$

$$\big(e_{u\lambda}(u,\lambda)[w,l], v\big)_{H^{-1},H_0^1} = 2\int_\Omega wlv, \qquad e_{\lambda\lambda}(u,\lambda) = 0 \tag{2.21c}$$

$$\big(e_{uu}(u,\lambda)[w,\eta], v\big)_{H^{-1},H_0^1} = \Big(h''(Du)[Dw, D\eta], Dv\Big)_{L^2}, \tag{2.21d}$$

for all $v \in H_0^1(\Omega)$.

Note that for any fixed $\lambda \in H^1(\Omega)$ and $u \in H_0^1(\Omega)$ we get

$$\big(e_u(u,\lambda)w, v\big)_{H^{-1},H_0^1} = \mu\big(Dw, Dv\big)_{L^2} + \big(h'_\gamma(Du)Dw, Dv\big)_{L^2} + 2\int_\Omega \lambda wv dx, \tag{2.22}$$

for all $v \in H_0^1(\Omega)$. Now let $a := 1 - \frac{1}{2\gamma}$ and $b := 1 + \frac{1}{2\gamma}$, and let us introduce the sets

$$\begin{gathered}\mathcal{A}^\gamma(u) := \big\{ \in \Omega : \gamma|Du(x)| \geq b\big\}; \qquad \mathcal{S}^\gamma(u) := \big\{x \in \Omega : a < \gamma|Du(x)| < b\big\};\\ \mathcal{I}^\gamma(u) := \big\{x \in \Omega : \gamma|Du(x)| \leq b\big\}\end{gathered} \tag{2.23}$$

and $t_1(u) := \frac{\gamma}{2}\big(\gamma|Du| - a\big) = \frac{\gamma}{2}\big(\gamma|Du| - 1 + \frac{1}{2\gamma}\big)$; $t_2(u) = \gamma|Du| - 1 - \frac{3}{2\gamma}$. For all $z \in H_0^1(\Omega)$, we get the following expressions for the derivatives of h_γ:

$$\begin{aligned} h'_\gamma(Du)^* Dz = {} & \chi_{\mathcal{A}^\gamma(u)}\left\{\frac{Dz}{|Du|} - \frac{\langle Du, Dz\rangle}{|Du|^3}Du\right\}\\ & + \chi_{\mathcal{S}^\gamma(u)}\left\{\frac{\gamma}{2}Dz + \gamma^2\big(\gamma|Du| - 1\big)\left[2\gamma^2 t_1(u)t_2(u) - 1\right]\frac{\langle Du, Dz\rangle}{|Du|^2}Du\right.\\ & \left. + \left[\frac{2\gamma - 1}{4\gamma} - \frac{\gamma t_1(u)t_2(u)}{2} + \frac{\gamma^3 t_1^2(u)t_2^2(u)}{2}\right]\left(\frac{Dz}{|Du|} - \frac{\langle Du, Dz\rangle}{|Du|^3}Du\right)\right\}\\ & + \chi_{\mathcal{I}^\gamma(u)}\big(\gamma Dz\big)\end{aligned} \tag{2.24}$$

and

$$\begin{aligned} h''_\gamma(Du)[Dp, Dz] = {} & \chi_{\mathcal{A}^\gamma(u)}\Phi(Du, Dp)Dz\\ & + \chi_{\mathcal{S}^\gamma(u)}\left\{\left[\frac{\gamma}{2}t_1(u)t_2(u)\Big(4\gamma^3|Du|\big(\gamma|Du| - 1\big) - \gamma^2 t_1(u)t_2(u) + 1\Big)\right.\right.\\ & \left. - \left(\gamma^3|Du|^2 - \gamma^2|Du| + \frac{1}{2} - \frac{1}{4\gamma}\right)\right]\Phi(Du, Dp)Dz\\ & \left. + 6\gamma^5 t_1(u)t_2(u)\frac{\langle Du, Dp\rangle(DuDu^T)}{|Du|^3}Dz\right\},\end{aligned} \tag{2.25}$$

where the operator

$$\Phi(Du, Dp) := \frac{3\langle Du, Dp\rangle(DuDu^T)}{|Du|^5} - \frac{(DpDu^T)}{|Du|^3} - \frac{(DuDp^T)}{|Du|^3} - \frac{\langle Du, Dp\rangle}{|Du|^3}.$$

We obtain the following property of the second derivative $e_{uu}(u, \lambda)$.

Lemma 2.2. *Let $p^* \in H_0^1(\Omega)$ be a solution to* (2.14d)-(2.14f)*, and assume that the hypotheses of Theorem 2.1 hold. Then there exist constants $L_1^\gamma, L_2^\gamma > 0$ only depend on γ, such that* *for any* $w, \eta \in H_0^1(\Omega)$

$$\left|\left\langle p^*, e_{uu}(u,\lambda)[w,\eta]\right\rangle_{H_0^1\times H^{-1}}\right| \leq L_1^\gamma \|p^*\|_{H_0^1}\|w\|_{H_0^1}\|\eta\|_{H_0^1}, \tag{2.26}$$

$$\left|\left\langle p^*, [e_{uu}(u,\lambda)-e_{uu}(\hat u,\lambda)][w,\eta]\right\rangle_{H_0^1\times H^{-1}}\right| \leq L_2^\gamma \|u-\hat u\|_{H_0^1}\|p^*\|_{H_0^1}\|w\|_{H_0^1}\|\eta\|_{H_0^1}, \tag{2.27}$$

for every $u, \hat u \in H_0^1(\Omega)$.

Proof. From the regularity conditions, we first have for every $w, \eta \in H_0^1(\Omega)$

$$\|h_\gamma''(Du)[Dw, D\eta]\|_{L^2} \leq \|h_\gamma''(Du)^* Dw\|_{L^s}\|D\eta\|_{L^r} = O\big(\|w\|_{H_0^1}\|\eta\|_{H_0^1}\big),$$

where $1/s + 1/r = 1$. Therefore $h_\gamma''(Du)[Dw, D\eta] \in \mathbb{L}^2(\Omega)$. It then follows

$$\left|\left\langle p^*, e_{uu}(u,\lambda)[w,\eta]\right\rangle_{H_0^1, H_0^{-1}}\right| = \left|\big(Dp^*, h_\gamma''(Du)[Dw, D\eta]\big)_{L^2}\right| = O\big(\|p^*\|_{H_0^1}\|w\|_{H_0^1}\|\eta\|_{H_0^1}\big).$$

From (2.26) and Lemma 2.1, we get (2.27). □

We define the cone of critical directions by

$$\mathcal{K}(\lambda^*) = \left\{ l \in H^1(\Omega) : l(x) \begin{cases} = 0 & \text{if} \quad \vartheta(x) \neq 0 \\ \geq 0 & \text{if} \quad \vartheta(x) = 0 \quad \text{and} \quad \lambda^*(x) = 0 \end{cases} \right\}. \tag{2.28}$$

Now let us state the second order optimality condition for problem (2.1). The proof goes along the lines of [21,22]. However, since in our case the control enters in a bilinear way and the PDE has a quasilinear structure, the proof has to be modified accordingly.

Theorem 2.3. *Let* (u^*, λ^*, p^*) *be a solution of the optimality system* (2.14) *and suppose that*

$$2\|w\|_{L^2}^2 + 2\beta\|l\|_{H^1}^2 + \int_\Omega \big(\big[h'(Du^*)Dw\big]_u w, Dp^*\big)dx + 4\int_\Omega wlp^* dx \geq \rho\|l\|_{H^1}^2, \tag{2.29}$$

for every pair $(w, l) \in H_0^1(\Omega) \times \mathcal{K}(\lambda^*)$, $(w, l) \neq (0, 0)$ *which satisfies the linearized equation:*

$$\mu\big(Dw, Dv\big)_{L^2} + \big(h_\gamma'(Du^*)Dw, Dv\big)_{L^2} + 2\int_\Omega l(u^*-f)v dx + 2\int_\Omega \lambda^* wv dx = 0, \forall v \in V. \tag{2.30}$$

Then there exist $\sigma > 0$ *and* $\tau > 0$ *such that*

$$J(u^*, \lambda^*) + \tau\|\lambda - \lambda^*\|_{H^1}^2 \leq J(u, \lambda), \tag{2.31}$$

for every feasible pair (u, λ) *such that* $u = G(\lambda)$ *and* $\|\lambda - \lambda^*\|_{H^1} \leq \sigma$.

Proof. Suppose that λ^* does not satisfy the growth condition (2.31). Then there exists a feasible sequence $\{\lambda_k\}_k \subset H^1(\Omega)$ such that

$$\|\lambda_k - \lambda^*\|_{H^1} < \frac{1}{k^2} \quad \text{and} \tag{2.32}$$

$$J(u^*, \lambda^*) + \frac{1}{k}\|\lambda - \lambda^*\|_{H^1}^2 > J(u_k, \lambda_k) = \mathcal{L}(u_k, \lambda_k, p^*) \quad \forall k, \tag{2.33}$$

where $u_k = G(\lambda_k)$ and $\mathcal{L}(u, \lambda, p) := \langle e(u, \lambda), p\rangle + J(u, \lambda)$. By setting $\rho_k = \|\lambda_k - \lambda^*\|_{H^1}$ and $\zeta_k = \frac{1}{\rho_k}(\lambda_k - \lambda^*)$ it follows that $\|\zeta_k\|_{H^1} = 1$ and therefore we may extract a subsequence, denoted the same, which converges to ζ weakly in $H^1(\Omega)$.

Step 1. By the mean value theorem we have

$$\begin{aligned} \mathcal{L}(u_k, \lambda_k, p^*) + \mathcal{L}_u(\nu_k, \lambda_k, p^*)(u^* - u_k) &= \mathcal{L}(u^*, \lambda_k, p^*) \\ &= \mathcal{L}(u^*, \lambda^*, p^*) + \rho_k \mathcal{L}_\lambda(u^*, \xi_k, p^*)\zeta_k \end{aligned}$$

where ν_k, ξ_k are points between u^* and u_k, λ^* and λ_k, respectively. From (2.33) and $J(u^*, \lambda^*) = \mathcal{L}(u^*, \lambda^*, p^*)$ it follows that

$$\mathcal{L}_\lambda(u^*, \xi_k, p^*)\zeta_k < \frac{1}{k}\|\lambda_k - \lambda^*\|_{H^1} + \frac{1}{\rho_k}\mathcal{L}_u(\nu_k, \lambda_k, p^*)(u^* - u_k). \tag{2.34}$$

By using again the mean value theorem for the last term on the first variable, we obtain

$$\begin{aligned}\mathcal{L}_u(\nu_k,\lambda_k,p^*)(u^*-u_k) =& J_u(\nu_k)(u^*-u_k) + \langle p^*, e_u(\nu_k,\lambda_k)(u^*-u_k)\rangle_{H_0^1,H^{-1}} \\ =& J_u(\nu_k)(u^*-u_k) + \langle p^*, e_u(u^*,\lambda_k)(u^*-u_k)\rangle_{H_0^1,H^{-1}} \\ &+ \langle p^*, e_{uu}(u^*,\lambda_k)(\nu_k-u^*)(u^*-u_k)\rangle_{H_0^1,H^{-1}} \\ &+ \big\langle p^*, \big(e_{uu}(\eta_k,\lambda_k)-e_{uu}(u^*,\lambda_k)\big)(\nu_k-u^*)(u^*-u_k)\big\rangle_{H_0^1,H^{-1}},\end{aligned}$$

where $\eta_k = u^* + t(\nu_k - u^*)$, for some $t \in [0,1]$. From (2.22) and the optimality system (2.14) it follows that

$$\begin{aligned}&\langle p^*, e_u(u^*,\lambda_k)(u^*-u_k)\rangle_{H_0^1,H^{-1}} \\ &\qquad= \langle p^*, e_u(u^*,\lambda^*)(u^*-u_k)\rangle_{H_0^1,H^{-1}} + 2\int_\Omega (\lambda_k-\lambda^*)(u^*-u_k)p^* \\ &\qquad= -J_u(u^*)(u^*-u_k) + 2\int_\Omega (\lambda_k-\lambda^*)(u^*-u_k)p^*.\end{aligned}$$

Hence, from (2.26) and (2.27) we get

$$\begin{aligned}\big|\mathcal{L}_u(\nu_k,\lambda_k,p^*)(u^*-u_k)\big| \le& \|J_u(\nu_k)-J_u(u^*)\|_{H^{-1}}\|u^*-u_k\|_{H_0^1} \\ &+ 2\|\lambda_k-\lambda^*\|_{L^3}\|u^*-u_k\|_{L^3}\|p^*\|_{L^3} \\ &+ L_1^\gamma\|p^*\|_{H_0^1}\|\nu_k-u^*\|_{H_0^1}\|u^*-u_k\|_{H_0^1} \\ &+ L_2^\gamma\|\nu_k-u^*\|_{H_0^1}^2\|p^*\|_{H_0^1}\|u^*-u_k\|_{H_0^1}.\end{aligned}$$

Due to the quadratic cost and the convergence $\zeta_k \rightharpoonup \zeta$, $\xi_k \to \lambda^*$ in $H^1(\Omega)$ and $u_k \to u^*$ in $H_0^1(\Omega)$, from (2.34) it follows that

$$\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta = \lim_{k\to\infty}\mathcal{L}_\lambda(u^*,\xi_k,p^*)\zeta_k \le 0.$$

On the other hand, since $\lambda_k(x) \ge 0$ a.e in Ω, it follows that

$$\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta_k = \rho_k\mathcal{L}_\lambda(u^*,\lambda^*,p^*)(\lambda_k-\lambda^*) \ge 0. \tag{2.35}$$

Since $\zeta_k \rightharpoonup \zeta$ one gets $\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta = \lim_{k\to\infty}\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta_k \ge 0$.
Thus, altogether we obtain that $\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta = 0$.

Step 2. Now we show that $\zeta \in \mathcal{K}(\lambda^*)$. The set

$$\big\{v \in H^1(\Omega) : v(x) \ge 0 \quad \text{if} \quad \vartheta(x) = 0 \quad \text{and} \quad \lambda^*(x) = 0\big\}$$

is convex and closed, hence it is weakly sequentially closed. Since λ_k is feasible, then for each k, ζ_k belongs to this set and, consequently, ζ also does. From (2.14i) it follows that $\vartheta(x)\zeta(x) \ge 0$ a.e in Ω, which implies

$$0 = \mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta = \beta\big(\lambda^*,\zeta\big)_{H^1} + \int_\Omega (u^*-f)p^*\zeta = \int_\Omega \vartheta\zeta = \int_\Omega |\vartheta\zeta|.$$

It follows that $\zeta(x) = 0$ if $\vartheta(x) \ne 0$ and therefore $\zeta \in \mathcal{K}(\lambda^*)$.

Step 3 ($\zeta = 0$). Using a Taylor expansion of the Lagrangian $\mathcal{L}$ at (u^*,λ^*,p^*) we have

$$\begin{aligned}\mathcal{L}(u_k,\lambda_k,p^*) =& \mathcal{L}(u^*,\lambda^*,p^*) + \rho_k\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta_k + \frac{\rho_k^2}{2}\mathcal{L}_{\lambda\lambda}(u^*,\lambda^*,p^*)\zeta_k^2 \\ &+ \rho_k\mathcal{L}_{u\lambda}(u^*,\lambda^*,p^*)(u_k-u^*)\zeta_k + \frac{1}{2}\mathcal{L}_{uu}(\nu_k,\lambda^*,p^*)(u_k-u^*)^2,\end{aligned} \tag{2.36}$$

where ν_k is an intermediate point between u_k and u^*. Therefore, thanks to the bilinear control structure,

$$\begin{aligned}\rho_k\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta_k &+ \frac{\rho_k^2}{2}\mathcal{L}_{\lambda\lambda}(u^*,\lambda^*,p^*)\zeta_k^2 + \rho_k\mathcal{L}_{u\lambda}(u^*,\lambda^*,p^*)(u_k-u^*)\zeta_k \\ &+ \frac{\rho_k^2}{2}\mathcal{L}_{uu}(u^*,\lambda^*,p^*)\left(\frac{u_k-u^*}{\rho_k}\right)^2 \\ = \mathcal{L}(u_k,\lambda_k,p^*) &- \mathcal{L}(u^*,\lambda^*,p^*) \\ &+ \frac{\rho_k^2}{2}\Big[\mathcal{L}_{uu}(u^*,\lambda^*,p^*) - \mathcal{L}_{uu}(\nu_k,\lambda^*,p^*)\Big]\left(\frac{u_k-u^*}{\rho_k}\right)^2.\end{aligned} \tag{2.37}$$

Moreover, from (2.33) it follows that

$$\mathcal{L}(u_k,\lambda_k,p^*) - \mathcal{L}(u^*,\lambda^*,p^*) < \frac{\rho_k^2}{k}. \tag{2.38}$$

From the differentiability of G, we have that $\|\frac{u_k-u^*}{\rho_k}\| = \frac{\|G(\lambda_k)-G(\lambda^*)\|_{H_0^1}}{\|\lambda_k-\lambda^*\|_{H^1}}$ is bounded. Hence, from $\lambda_k \to \lambda^*$, $\|\zeta_k\|_{H^1} = 1$ and by (2.27) we obtain

$$\begin{aligned}&\left|\Big[\mathcal{L}_{uu}(u^*,\lambda^*,p^*) - \mathcal{L}_{uu}(\nu_k,\lambda^*,p^*)\Big]\left(\frac{u_k-u^*}{\rho_k}\right)^2\right| \\ &\quad\le L_2^\gamma\|p^*\|_{H_0^1}\|u^*-u_k\|_{H_0^1}\left\|\frac{u_k-u^*}{\rho_k}\right\|_{H_0^1}^2 \underset{k\to\infty}{\longrightarrow} 0.\end{aligned} \tag{2.39}$$

From (2.37) it follows that

$$\begin{aligned}\liminf_{k\to\infty}\mathcal{L}_{\lambda\lambda}(u^*,\lambda^*,p^*)\zeta_k^2 &+ \liminf_{k\to\infty}\mathcal{L}_{uu}(u^*,\lambda^*,p^*)\left(\frac{u_k-u^*}{\rho_k}\right)^2 \\ &+ 2\liminf_{k\to\infty}\frac{1}{\rho_k}\mathcal{L}_{u\lambda}(u^*,\lambda^*,p^*)(u_k-u^*)\zeta_k \\ \le 2\limsup_{k\to\infty}\frac{1}{\rho_k^2}\big[\mathcal{L}(u_k,\lambda_k,p^*) &- \mathcal{L}(u^*,\lambda^*,p^*)\big] - 2\liminf_{k\to\infty}\frac{1}{\rho_k}\mathcal{L}_\lambda(u^*,\lambda^*,p^*)\zeta_k.\end{aligned}$$

Since $\mathcal{L}_{\lambda\lambda}(u^*,\lambda^*,p^*)\zeta_k^2 = 2\beta\|\zeta_k\|_{H^1}^2$ is weakly lower semi-continuous and from (2.35), (2.38), the last relation implies

$$\begin{aligned}\mathcal{L}_{\lambda\lambda}(u^*,\lambda^*,p^*)\zeta^2 &+ \liminf_{k\to\infty}\mathcal{L}_{uu}(u^*,\lambda^*,p^*)\left(\frac{u_k-u^*}{\rho_k}\right)^2 \\ &+ 2\liminf_{k\to\infty}\mathcal{L}_{u\lambda}(u^*,\lambda^*,p^*)\left(\frac{u_k-u^*}{\rho_k}\right)\zeta_k \le 2\lim_{k\to\infty}\frac{1}{k} = 0.\end{aligned} \tag{2.40}$$

Let us denote by ϑ_{ζ_k} the solution of (2.30) associated with ζ_k. Since $\zeta_k \rightharpoonup \zeta$ in $H^1(\Omega)$ and $\|\zeta_k\|_{H^1} = 1$ one gets that $\zeta_k \to \zeta$ in $L^p(\Omega)$, for all $p \in [1,\infty)$. Hence, from (2.30) and the continuous invertibility of $e_u(u^*,\lambda^*)$, we have $\vartheta_{\zeta_k} \to \vartheta_\zeta$.
Besides,

$$\begin{aligned}\mathcal{L}_{uu}(u^*,\lambda^*,p^*)\left(\frac{u_k-u^*}{\rho_k}\right)^2 &= \mathcal{L}_{uu}(u^*,\lambda^*,p^*)\left(\frac{G(\lambda_k)-G(\lambda^*)}{\|\lambda_k-\lambda^*\|} - \vartheta_{\zeta_k}\right)^2 \\ + 2\mathcal{L}_{uu}(u^*,\lambda^*,p^*)&\left(\frac{G(\lambda_k)-G(\lambda^*)}{\|\lambda_k-\lambda^*\|} - \vartheta_{\zeta_k},\vartheta_{\zeta_k}\right) + \mathcal{L}_{uu}(u^*,\lambda^*,p^*)(\vartheta_{\zeta_k})^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{u\lambda}(u^*,\lambda^*,p^*)\left(\frac{u_k-u^*}{\rho_k}\right)\zeta_k =& \mathcal{L}_{u\lambda}(u^*,\lambda^*,p^*)\left(\frac{G(\lambda_k)-G(\lambda^*)}{\|\lambda_k-\lambda^*\|} - \vartheta_{\zeta_k}\right)\zeta_k \\ &+ \mathcal{L}_{u\lambda}(u^*,\lambda^*,p^*)(\vartheta_{\zeta_k}\zeta_k).\end{aligned}$$

Note that ϑ_{ζ_k} also corresponds to the derivative of the control-to-state mapping G at λ^* in direction ζ_k. From the differentiability of G, it follows that $\frac{G(\lambda_k)-G(\lambda^*)}{\|\lambda_k-\lambda^*\|} - \vartheta_{\zeta_k} \underset{k\to\infty}{\longrightarrow} 0$. Due to the continuity of the bilinear form $\mathcal{L}_{uu}(u^*, \lambda^*, p^*)$, since $\vartheta_{\zeta_k} \to \vartheta_\zeta$ and from (2.14g-2.14i), we get

$$\mathcal{L}_{\lambda\lambda}(u^*, \lambda^*, p^*)\zeta^2 + 2\mathcal{L}_{u\lambda}(u^*, \lambda^*, p^*)(\vartheta_\zeta \zeta) + \mathcal{L}_{uu}(u^*, \lambda^*, p^*)\vartheta_\zeta^2 \leq 2 \lim_{k\to\infty} \frac{1}{k} = 0.$$

Since $\zeta \in \mathcal{K}(\lambda^*)$, from (2.29) it then follows that $(\zeta, \vartheta_\zeta) = 0$.

Step 4: Finally, from $\vartheta_{\zeta_k} \to \vartheta_\zeta = 0$, (2.29), (2.35), (2.38) we have

$$\lim_{k\to\infty} \sup \rho \|\zeta_k\|_{H^1}^2 \leq \lim_{k\to\infty} \sup \mathcal{L}_{\lambda\lambda}(u^*, \lambda^*, p^*)\zeta_k^2 \leq 2 \lim_{k\to\infty} \frac{1}{k} = 0.$$

Hence, $\zeta_k \to 0$ in $H^1(\Omega)$, which is in contradiction to $\|\zeta_k\|_{H^1} = 1$. □

3. Discretization and numerical treatment

In this section we present a numerical strategy for the solution of problem (2.20). We start by explaining how the domain is discretized using finite differences and introduce the resulting discrete operators. Due to the size of the problem, an overlapping Schwarz domain decomposition strategy is considered, where the transmission conditions between subdomains are determined in an optimized way. The resulting subdomain finite-dimensional nonlinear systems are then solved by using a semismooth Newton method, for which local superlinear convergence is proved. A further modification of the semismooth Newton algorithm is introduced in order to get a global convergent behaviour.

3.1. **Discretization schemes.** For the image domain, we use a finite differences scheme on a uniform mesh and consider the problem (2.20) on the domain $\Omega := [0, (w-1)h] \times [0, (l-1)h]$, where h denotes the mesh step size, and $w, l \in \mathbb{N}^*$ depend on the resolution of the input data. In practice, w and l are width and length of the input images $f, u^\dagger$ in pixels. In what follows, the notation $\tilde{u}, \tilde{q}, \tilde{p}, \tilde{z}, \tilde{\lambda}$ is used for the discretized variables that approximate u, q, p, z, λ and F_h, Div_h, Δ_h are used for the discrete approximations of F, Div, Δ, respectively.

In order to approximate the state and adjoint variables, as well as their derivatives, we consider a modified finite differences scheme (see [23]). We define the following grid domains:

$$\begin{aligned}
\Omega_h &= \{x_{ij} := ((i-1)h, (j-1)h) | i = 1, \ldots, w; j = 1, \ldots, l\}; \\
\Omega_h^1 &= \{x_{ij} := ((i-0.5)h, (j-1)h) | i = 1, \ldots, w; j = 1, \ldots, l\}; \\
\Omega_h^2 &= \{x_{ij} := ((i-1)h, (j-0.5)h) | i = 1, \ldots, w; j = 1, \ldots, l\}.
\end{aligned}$$

and the corresponding spaces of grid functions:

$$\begin{aligned}
U_h &= \{u_{ij} := u(x_{ij}) | x_{ij} \in \Omega_h; \quad u_{i0} = u_{0j} = 0; \quad 1 \leq i \leq w, \quad 1 \leq j \leq l\}; \\
\Lambda_h &= \{\lambda_{ij} := \lambda(x_{ij}) | x_{ij} \in \Omega_h; \quad 1 \leq i \leq w, \quad 1 \leq j \leq l\}; \\
D_u^1 &= \{u_{ij} := u(x_{ij}) | x_{ij} \in \Omega_h^1; \quad u_{i0} = u_{0j} = 0; \quad 1 \leq i \leq w, \quad 1 \leq j < l\}; \\
D_u^2 &= \{u_{ij} := u(x_{ij}) | x_{ij} \in \Omega_h^2; \quad u_{i0} = u_{0j} = 0; \quad 1 \leq i < w, \quad 1 \leq j \leq l\}.
\end{aligned}$$

Therefore, $\tilde{u}, \tilde{p} \in U_h$, $\tilde{\lambda} \in \Lambda_h$ and $\tilde{q}, \tilde{z} \in D_u^1 \times D_u^2$. We define the operator D_h as follows:

$$D_h : \Lambda_h \longrightarrow D_u^1 \times D_u^2; \quad (D_h v)_{i,j} = \big((D_{h_{x_1}} v)_{i,j}, (D_{h_{x_2}} v)_{i,j}\big)$$

where $D_{h_{x_1}}$ and $D_{h_{x_2}}$ are computed by forward differences of the "inner points"

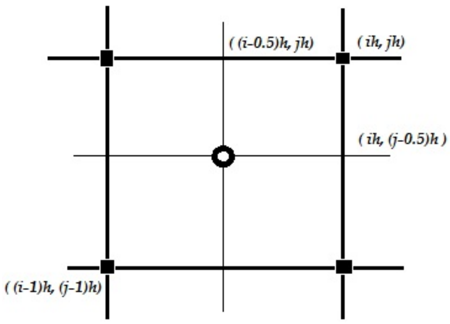
$$(D_{h_{x_1}} v)_{i,j} := \frac{v_{i+1,j} - v_{i,j}}{h}; \quad (D_{h_{x_2}} v)_{i,j} := \frac{v_{i,j+1} - v_{i,j}}{h}; \quad 1 \leq i < w-1, 1 \leq j < l-1.$$

The discrete Laplacian $\Delta_h : \Lambda_h \to \Lambda_h$ is computed by using a classical five point stencil. For the Neumann boundary conditions $\dfrac{\partial u}{\partial n_i} = \dfrac{\partial p}{\partial n_i} = \dfrac{\partial \lambda}{\partial n_i} = 0$, $(i = 1, 2)$ we get

$$\tilde{u}_{0,j} = \tilde{u}_{2,j}; \quad \tilde{u}_{w+1,j} = \tilde{u}_{w-1,j} \quad (1 \le j \le l); \quad \tilde{u}_{i,2} = \tilde{u}_{i,0}; \quad \tilde{u}_{i,l+1} = \tilde{u}_{i,l-1} \quad (1 \le i \le w)$$
$$\tilde{p}_{0,j} = \tilde{p}_{2,j}; \quad \tilde{p}_{w+1,j} = \tilde{p}_{w-1,j} \quad (1 \le j \le l); \quad \tilde{p}_{i,2} = \tilde{p}_{i,0}; \quad \tilde{p}_{i,l+1} = \tilde{p}_{i,l-1} \quad (1 \le i \le w)$$
$$\tilde{\lambda}_{0,j} = \tilde{\lambda}_{2,j}; \quad \tilde{\lambda}_{w+1,j} = \tilde{\lambda}_{w-1,j} \quad (1 \le j \le l); \quad \tilde{\lambda}_{i,2} = \tilde{\lambda}_{i,0}; \quad \tilde{\lambda}_{i,l+1} = \tilde{\lambda}_{i,l-1} \quad (1 \le i \le w).$$

The discrete divergence operator $\mathrm{Div}_h : D_u^1 \times D_u^2 \to U_h$ is computed by using backward differences $\tilde{q} = (\tilde{q}^1, \tilde{q}^2) \in D_u^1 \times D_u^2$

$$(\mathrm{Div}_h \tilde{q})_{i,j} = \frac{\tilde{q}^1_{i,j} - \tilde{q}^1_{i-1,j}}{h} + \frac{\tilde{q}^2_{i,j} - \tilde{q}^2_{i,j-1}}{h}.$$



- ■ : Points for $\tilde{u}$, $\tilde{p}$, $\tilde{\lambda}$, Δ_h, $D_h q$, $D_h z$;
- ○ : Points for $D_h\tilde{u}$, $D_h\tilde{p}$, $D_h\tilde{\lambda}$, q, z.

FIGURE 3.1. Mesh structure of the discretization scheme

Accordingly, we define the approximation operator $F_h : H_h \to H_h'$, where $H_h = U_h \times (D_u^1 \times D_u^2) \times U_h \times (D_u^1 \times D_u^2) \times \Lambda_h$ and $H_h' = U_h \times (D_u^1 \times D_u^2) \times U_h \times (D_u^1 \times D_u^2) \times U_h$, and for $\tilde{\mathbf{y}} = (\tilde{u}, \tilde{q}, \tilde{p}, \tilde{z}, \tilde{\lambda}) \in H_h$, we have the equation

$$F_h(\tilde{\mathbf{y}}) = \begin{pmatrix} -\mu\Delta_h\tilde{u} - \mathrm{Div}_h\, \tilde{q} + 2\tilde{\lambda}.(\tilde{u} - f) \\ h_\gamma(D_h\tilde{u}) - \tilde{q} \\ -\mu\Delta_h\tilde{p} - \mathrm{Div}_h\, \tilde{z} + 2\tilde{\lambda}.\tilde{p} + 2(\tilde{u} - u^\dagger) \\ h'_\gamma(D_h\tilde{u})^* D_h\tilde{p} - \tilde{z} \\ -\beta\Delta_h\tilde{\lambda} + \beta\tilde{\lambda} + (\tilde{u} - f).\tilde{p} - \max\big(0, -\beta\Delta_h\tilde{\lambda} + (\tilde{u} - f).\tilde{p}\big) \end{pmatrix} = 0. \tag{3.1}$$

Above, we used the notation $u.v$ to present the grid function $(uv)_{ij} = u_{ij}v_{ij}$ for all $u, v \in \Lambda_h$ or $u, v \in D_u^k$ $(k = 1, 2)$. Hereafter, the notations $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ stand for the Euclidian product and norm in $\mathbb{R}^n$, respectively. Besides, for $q = (q^1, q^2), z = (z^1, z^2) \in D_u^1 \times D_u^2$, we denote $(q, z)_{D_u^1 \times D_u^2} := \langle q^1, z^1\rangle + \langle q^2, z^2\rangle$.

3.2. **Schwarz domain decomposition methods.** The nonlinear system (3.1), arising from the discretization of (2.20), is of large scale nature, involving the solution of three coupled PDEs per each training pair of images. Even for the case of a single training pair, this task cannot be performed on a desktop computer. In the case of larger training sets, the problem becomes much harder, not to mention the increasingly high resolution of the images at hand.

To tackle this problem, we consider the application of Schwarz domain decomposition methods for solving the resulting optimality system. Since our aim is to set up a parallel method based on domain decomposition, we focus on additive Schwarz methods. Once the domain is decomposed, the nonlinear optimality system is solved in each subdomain. Such an approach was considered, for instance, in [9] for the pure total variation denoising problem.

It is well-known that the convergence rate of the Schwarz method is dependent on the size of the overlapping area. In order to improve the convergence rate, a modified version of the method was proposed in [13, 24]. To illustrate the main idea, consider the following coupled linear system with an optimality system type structure:

$$\begin{aligned} -\Delta u + \eta u &= f + \theta p \quad \text{in} \quad \Omega, & u &= 0 \quad \text{on} \quad \partial\Omega; \\ -\Delta p + \eta p &= -(u - u_d) \quad \text{in} \quad \Omega, & p &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\theta, \eta > 0$. The so-called optimized Schwarz method (with two subdomains) is as follows: For $k \geq 0$; $i, j \in \{1, 2\}$, $i \neq j$

$$\begin{cases} -\Delta u_i^{k+1} + \eta u_i^{k+1} = f + \mu p_i^{k+1} \quad \text{in} \quad \Omega_i; \\ u_i^{k+1}|_{\partial\Omega} = 0; \quad (\alpha_i + \partial_{\vec{n}}) u_i^{k+1}|_{\Gamma_i} = (\alpha_i + \partial_{\vec{n}}) u_j^k|_{\Gamma_i}; \\ -\Delta p_i^{k+1} + \eta p_i^{k+1} = -(u_i^{k+1} - u_d) \quad \text{in} \quad \Omega_i; \\ p_i^{k+1}|_{\partial\Omega} = 0; \quad (\alpha_i + \partial_{\vec{n}}) p_i^{k+1}|_{\Gamma_i} = (\alpha_i + \partial_{\vec{n}}) p_j^k|_{\Gamma_i}, \end{cases}$$

where the transmission parameters α_1, α_2 are approximated as follows (by zero order approximations)

$$\alpha_1 = \sqrt{\eta}; \quad \alpha_2 = -\sqrt{\eta}.$$

In order to obtain the formulas for the transmission parameters of the optimized Schwarz method, we consider the equations for u and p in the optimality system (in strong form) as a coupled system

$$\begin{aligned} -\mu\Delta u - \operatorname{Div}[h_\gamma(Du)] + 2\lambda(u - f) &= 0, \\ -\mu\Delta p - \operatorname{Div}[h'_\gamma(Du)^* Dp] + 2\lambda p + 2(u - u^\dagger) &= 0. \end{aligned}$$

By skipping the regularization terms, we get again the linear coupled system as in [24] with $\mu = 1$. In addition, we consider the equation

$$-\beta\Delta\lambda + \beta\lambda + (u - f)p = 0$$

for the functional parameter λ. We use the common forms of transmission conditions on Γ_1, Γ_2 in optimized Schwarz method as follows

$$\begin{aligned} \Big(\frac{\partial}{\partial\vec{n}} + S_{v_1}^{(u^k,p^k,\lambda^k)}\Big) v_1^{k+1}(\cdot, x_2) &= \Big(\frac{\partial}{\partial\vec{n}} + S_{v_1}^{(u^k,p^k,\lambda^k)}\Big) v_2^k(\cdot, x_2) \quad \text{on} \quad \Gamma_1; \\ \Big(\frac{\partial}{\partial\vec{n}} + S_{v_2}^{(u^k,p^k,\lambda^k)}\Big) v_2^{k+1}(\cdot, x_2) &= \Big(\frac{\partial}{\partial\vec{n}} + S_{v_2}^{(u^k,p^k,\lambda^k)}\Big) v_1^k(\cdot, x_2) \quad \text{on} \quad \Gamma_2, \end{aligned} \tag{3.2}$$

where the transmission parameters are chosen in a similar way as for the coupled (linear) system (see [24]):

$$S_{u_1}^{(u^k,p^k,\lambda^k)} = S_{p_1}^{(u^k,p^k,\lambda^k)} = \sqrt{\frac{2\lambda_1^n}{\mu}}; \quad S_{u_2}^{(u^k,p^k,\lambda^k)} = S_{p_2}^{(u^k,p^k,\lambda^k)} = -\sqrt{\frac{2\lambda_2^k}{\mu}};$$
$$S_{\lambda_1}^{(u^k,p^k,\lambda^k)} = 1; \quad S_{\lambda_2}^{(u^k,p^k,\lambda^k)} = -1.$$

3.3. **Semismooth Newton method.** The optimality system (3.1) has a nonlinear nonsmooth structure. Because of this, a Newton method cannot be directly applied. However, the nonsmooth functions involved, in particular the max operator, have additional properties, which allow to define a generalized Newton step for the solution of the system.

Definition 3.1. *Let X, Z be Banach spaces and $D \subset X$ be an open set. The mapping $F : D \to Z$ is called Newton differentiable on an open set $U \subset D$ if there exists a mapping $G : U \to \mathcal{L}(X, Z)$ such that*

$$\lim_{h\to 0} \frac{\|F(x+h) - F(x) - G(x+h)h\|_Z}{\|h\|_X} = 0, \quad h \in X$$

for every $x \in U$. G is called generalized derivative of F.

We also refer to [16, 17] for a chain rule for Newton differentiable functions.

Lemma 3.1. *Let $F : X \to Z$ be a Newton differentiable operator with generalized derivative G; x^* be a solution of equation $F(x) = 0$ and $U \subset X$ be an open neighborhood containing x^*. If for every $y \in U$, $\|[G(y)]^{-1}\|_{\mathcal{L}(X,Z)}$ is bounded, then the Newton iterations*

$$x_{k+1} = x_k - G^{-1}(x_k)F(x_k)$$

converge superlinearly to x^*, provided that $\|x_0 - x^*\|_X$ is sufficiently small.

In particular, it has been proved (see, e.g., [17]) that the mapping $\max(0, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is Newton differentiable with generalized derivative $G_m : \mathbb{R}^n \to \mathcal{L}\big(\mathbb{R}^n, \mathbb{R}^n\big)$ given by

$$(G_m(y))_i = \begin{cases} 1 & \text{if} \quad y_i > 0, \\ 0 & \text{if} \quad y_i \leq 0 \end{cases} .$$

The operator F_h in (3.1) is therefore Newton differentiable and its generalized derivative $\mathcal{G}_F : H_h \mapsto \mathcal{L}(H_h, H_h')$ is given by

(3.3) $\quad \mathcal{G}_{F_h}(\mathbf{y})\delta_{\mathbf{y}} =$

$$\begin{pmatrix} (2\lambda\mathbf{I} - \mu\Delta_h)\delta_u - \mathrm{Div}_h\delta_q + 2(u-f)\delta_\lambda \\ h_\gamma'(D_hu)D_h\delta_u - \delta_q \\ 2\delta_u + (2\lambda\mathbf{I} - \mu\Delta_h)\delta_p - \mathrm{Div}_h\delta_z + 2p\delta_\lambda \\ \big(h_\gamma'(D_hu)^*D_hp\big)_u\delta_u + h_\gamma'(D_hu)D_h\delta_p - \delta_z \\ p\delta_u + (u-f)\delta_p + \beta(\mathbf{I} - \Delta_h)\delta_\lambda - G_m\big((u-f).p - \beta\Delta_h\lambda\big)\big(p\delta_u + (u-f)\delta_p - \beta\Delta_h\delta_\lambda\big) \end{pmatrix}$$

where $\delta_{\mathbf{y}} = (\delta_u, \delta_q, \delta_p, \delta_z, \delta_\lambda)$ and $\mathbf{I}$ stands for the identify. The semi-smooth Newton step is then given by

$$\mathcal{G}_{F_h}(\mathbf{y}_k)\delta_{\mathbf{y}} = -F_h(\mathbf{y}_k), \quad \mathbf{y}_{k+1} = \mathbf{y}_k + \delta_{\mathbf{y}}, \tag{3.4}$$

where F and $\mathcal{G}_{F_h}$ are defined in (3.1) and (3.3), respectively.

For the convergence analysis, we also assume that there exists an optimal solution $(u^*, \lambda^*) \in U_h \times \Lambda_h$, with $\lambda^* \geq 0$ on Ω_h. The second order condition in Theorem 2.3 ensures that a solution of first order system is also solution of the optimization problem. However, to consider the convergence of the semi-smooth Newton method, we need the following stronger assumption: There exists $\rho > 0$ such that

$$\begin{aligned} 2\|w\|^2 + \beta(\|l\|^2 + \|D_hl\|^2) + \big(h''(D_hu^*)[D_hw, D_hw], D_hp^*\big)_{D_u^1 \times D_u^2} \\ + 4\langle w\, l, p^*\rangle \geq \rho(\|l\|^2 + \|D_hl\|^2), \end{aligned} \tag{3.5}$$

for every pair $(w, l) \in U_h \times \Lambda_h$ that satisfies

$$-\mu\Delta_hw - \mathrm{Div}_h\big(h_\gamma'(D_hu^*)D_hw\big) + 2(u^* - f)\ l + 2\lambda^*\ w = 0.$$

Instead of $\mathcal{K}$ in (2.28), we consider the discrete cone

$$\tilde{\mathcal{K}} = \{l_{ij} = l(x_{ij}) \in \Lambda_h : \quad l_{ij} \geq 0; \quad l_{ij} \not\equiv 0 \quad \forall x_{ij} \in \Omega_h\}.$$

From the monotonicity of $h_\gamma(z)$ it follows that $h_\gamma'(z)$ is positive semidefinite. Hence, the mapping $W : \Lambda_h \to \Lambda_h$ defined by $W(w) = -\mathrm{Div}_h\big(h_\gamma'(D_hu^*)D_hw\big)$ $(u \in U_h)$ is also positive semidefinite. Now we consider the mapping $e_u(u, \lambda) \in \mathcal{L}(U_h, U_h)$ defined by

$$e_u(u, \lambda)w = -\mu\Delta_hw - \mathrm{Div}_h\big(h_\gamma'(D_hu)D_hw\big) + 2\lambda\ w \quad \forall w \in U_h.$$

We have $\langle e_u(u, \lambda)w, w\rangle \geq \langle(2\lambda\mathbf{I} - \mu\Delta_h)w, w\rangle \quad \forall w \in U_h$. Besides, for all $u \in U_h$ and $\lambda \in \tilde{\mathcal{K}}$, the operator $(2\lambda\mathbf{I} - \mu\Delta_h) : U_h \to U_h$ is positive definite. It follows that $e_u(u, \lambda)$ is positive definite and hence invertible. Moreover, from the last inequality we see that for fixed $u \in U_h$ and $\lambda \in \mathcal{K}$, there exists $C = C(\lambda) > 0$ (only dependent on λ) such that for every $\xi \in U_h$, the equation

$$e_u(u, \lambda)w = -\mu\Delta_hw - \mathrm{Div}_h\big(h_\gamma'(D_hu)D_hw\big) + 2\lambda\ w = \xi$$

has a unique solution $w \in U_h$ which satisfies $\|w\|_{U_h} \leq C\|\xi\|_{U_h}$.

In order to consider the convergence of the method, we need C independent of λ. We use an additional assumption.

Assumption 3.1. *There exists a neighbourhood $V(\lambda^*)$ of the optimal parameter λ^* and a constant $C > 0$ (independent of u and λ) such that, for all $(u, \lambda) \in U_h \times V(\lambda^*)$ and for any $\xi \in U_h$, there exists unique solution $w \in U_h$ of $e_u(u, \lambda)w = \xi$ and $\|w\|_V \leq C\|\xi\|$.*

If a pair $(w, l) \in U_h \times \Lambda_h$ satisfies the equation

$$e_u(u, \lambda)w + e_\lambda(u, \lambda)l = -\mu\Delta_h w - \mathrm{Div}_h\big(h'_\gamma(D_h u)D_h w\big) + 2\lambda.w + 2(u - f)\,l = 0,$$

then $\|w\| \leq C_1(u, \lambda)\|l\|$, where $C_1(u, \lambda) > 0$ is dependent on (u, λ). If Assumption 3.1 holds and we only consider u in a bounded neighborhood of u^*, the last estimate yields $\|w\| \leq C_1\|l\|$ for some $C_1 > 0$ and for all $w \in U_h, l \in \Lambda_h$ satisfy $e_u(u, \lambda)w + e_\lambda(u, \lambda)l = 0$.

Lemma 3.2. *Let Assumption 3.1 hold and $(u, \lambda) \in V_1(u^*) \times V(\lambda^*)$, where $V_1(u^*)$ is a bounded neighborhood of u^*, $V(\lambda^*)$ is the neighborhood mentioned in Assumption 3.1. Then there exists a constant $\theta > 0$ such that for all solution $(w, l) \in U_h \times \Lambda_h$ of the equation $e_u(u, \lambda)w + e_\lambda(u, \lambda)l = 0$, we have the estimate $\|w\| \leq \theta\|l\|$.*

It is easy to see that in case we consider $\lambda \in \mathbb{R}$ and $\lambda^* > 0$ (the case $\lambda^* = 0$ is trivial), there always exist a bounded neighborhood $V_1(u^*)$ of u^* and a constant $r > 0$ such that all properties in Assumption 3.1 and Lemma 3.2 hold for all $(u, \lambda) \in V_1(u^*) \times (\lambda^* - r, \lambda^* + r)$ without any additional assumption.

Theorem 3.1. *Let Assumption 3.1 and condition* (3.5) *hold. Then the semismooth Newton method applied to* (3.1)*, with generalized derivative $\mathcal{G}_{F_h}$ defined by* (3.3)*, converges locally superlinearly to a solution $\mathbf{y}^* = (u^*, q(u^*), p(u^*, \lambda^*), z(u^*, \lambda^*), \lambda^*, \mu(\lambda^*))$, provided that $\|\mathbf{y}_0 - \mathbf{y}^*\|$ is sufficiently small.*

Proof. At step $k \geq 0$, we denote $A_k := \{x_{ij} \in \Omega_h : (u - f).p - \beta\Delta_h\lambda > 0\}$, $I_k := \Omega_h \setminus A$; F_h^i are components of right-hand side, $i = 1, .., 5$. The 5^{th} equation of the system (3.4) can be expressed as

$$\begin{cases} \chi_{A_k}\beta\delta_\lambda = \chi_{A_k}F_h^5 \\ \chi_{I_k}\big\{p.\delta_u + (u - f).\delta_p + \beta(\mathbf{I} - \Delta_h)\delta_\lambda\big\} = \chi_{I_k}F_h^5. \end{cases}$$

By a short computation, we can write (3.4) in equivalent form as follows

$$\{(2\lambda_k - \mu\Delta_h) - \mathrm{Div}_h[h'_\gamma(D_h u_k)D_h]\}\delta_u + 2(u_k - f).\delta_\lambda = f_1 \tag{3.6a}$$

$$\begin{aligned} &\big\{2\mathbf{I} - \mathrm{Div}_h\big[\big(h'_\gamma(D_h u_k)^* D_h p_k\big)_u\big]\big\}\delta_u + \\ &\quad + \big\{(2\lambda_k - \mu\Delta_h) - \mathrm{Div}_h[h'_\gamma(D_h u_k)D_h]\big\}\delta_p + 2p_k.\delta_\lambda = 2f_2 \end{aligned} \tag{3.6b}$$

$$\chi_{I_k}\big\{p_k.\delta_u + (u_k - f).\delta_p + \beta(\mathbf{I} - \Delta_h)\delta_\lambda\big\} = \chi_{I_k}\beta(\mathbf{I} - \Delta_h)f_3 \tag{3.6c}$$

$$\chi_{A_k}\delta_\lambda = \chi_{A_k}f_4 \tag{3.6d}$$

where $f_1 = F_h^1 - \mathrm{Div}_h F_h^2$; $f_2 = \frac{1}{2}(F_h^3 - \mathrm{Div}_h F_h^4)$; $f_3 = \beta^{-1}(\mathbf{I} - \Delta_h)^{-1}F_h^5$ and $f_4 = \beta^{-1}F_h^5$.
For a fixed grid step size $h > 0$, we easily verify that there exist some constants m_1, m_2, m_3, m_4, $m_5 > 0$ such that $\|f_1\| \leq m_1\|F_h^1\| + m_2\|F_h^2\|_{D_u^1 \times D_u^2}$, $\|f_2\| \leq m_3\|F_h^3\| + m_4\|F_h^4\|_{D_u^1 \times D_u^2}$ and $\|f_3\| \leq m_5\|F_h^5\|$.
We use the techniques as in [21, 22] to show that there exists a neighborhood $V(u^*, \lambda^*, p^*)$ such that with any $(u, \lambda, p) \in V(u^*, \lambda^*, p^*)$ the system (3.4) is solvable for every right-hand side F_h^i. We write (3.6a), (3.6d) in form

$$E(\delta_u, \delta_\lambda) = \begin{pmatrix} e_u(u_k, \lambda_k)\delta_u + e_\lambda(u_k, \lambda_k)\delta_\lambda - f_1 \\ \chi_A(\delta_\lambda - f_4) \end{pmatrix} = 0, \text{ where } E : U_h \times \Lambda_h \to U_h \times \Lambda_h\big|_{A_k}.$$

We have $\ker(E') = \{(v, l) \in U_h \times \Lambda_h : \chi_A v = 0; e_u(u_k, \lambda_k)v + e_\lambda(u_k, \lambda_k)l = 0\}$.
To show the existence and uniqueness of a solution to (3.6), let us introduce the following auxiliary problem

$$\begin{aligned} \min \mathcal{J}_A(\delta_u, \delta_\lambda) = &\|\delta_u - f_2\|^2 + \beta\|\chi_{I_k}(\delta_\lambda - f_3)\|^2 + \beta\|\chi_{I_k}[D_h(\delta_\lambda - f_3)]\|^2 \\ &+ \frac{1}{2}\big\langle e_{uu}[\delta_u]^2, p_k\big\rangle + \big\langle e_{u\lambda}[\delta_u, \delta_\lambda], p_k\big\rangle \\ \text{subject to} \quad & E(\delta_u, \delta_\lambda) = 0. \end{aligned} \tag{3.7}$$

We consider the Lagrangian

$$\begin{aligned}\mathcal{L}^A(\delta_u,\delta_\lambda,\delta_p,\psi) &= \mathcal{J}_A(\delta_u,\delta_\lambda) + \langle \psi, \chi_A(\delta_\lambda - f_4)\rangle\big|_{A_k} \\ &\quad + \langle \delta_p, e_u(u_k,\lambda_k)\delta_u + e_\lambda(u_k,\lambda_k)\delta_\lambda - f_1\rangle.\end{aligned} \tag{3.8}$$

Now it is not difficult to show that (3.6) is the optimality condition for problem (3.7). The Hessian of $\mathcal{J}_A$ is determined by

$$\begin{aligned}\mathcal{J}_A''[\delta_u,\delta_\lambda]^2 &= 2\|\delta_u\|^2 + 2\beta\big(\|\chi_{I_k}\delta_\lambda\|^2 + \|\chi_{I_k}(D_h\delta_\lambda)\|^2_{D_u^1\times D_u^2}\big) \\ &\quad + \big([h_\gamma'(D_h u_k)^* D_h\delta_u]_u\delta_u, D_h p_k\big)_{D_u^1\times D_u^2} + 4\langle p_k.\delta_u,\delta_\lambda\rangle.\end{aligned}$$

For every $(\delta_u,\delta_\lambda)\in\ker(E')$, we have $\chi_{A_k}\delta_\lambda = 0$. Hence

$$\begin{aligned}\mathcal{J}_A''[\delta_u,\delta_\lambda]^2 &\geq 2\|\delta_u\|^2 + 2\beta\big(\|\delta_\lambda\|^2 + \|D_h\delta_\lambda\|^2_{D_u^1\times D_u^2}\big) \\ &\quad + \big([h_\gamma'(D_h u_k)^* D_h\delta_u]_u\delta_u, D_h p_k\big)_{D_u^1\times D_u^2} + 4\langle p_k.\delta_u,\delta_\lambda\rangle.\end{aligned}$$

By Lemma 2.1 and Assumption 3.1, there is a neighborhood of the solution such that inside it, $e_u(u,\lambda)^*$ is surjective, invertible and $e_{uu}(u)$ is Lipschitz continuous. Hence, from (3.5) there exists a neighborhood $V(u^*,\lambda^*,p^*)$ of the solution and a constant $\rho>0$, such that for all $(u,\lambda,p)\in V(u^*,\lambda^*,p^*)$, the estimates

$$\begin{aligned}2\|v\|^2 + 2\beta(\|l\|^2 + \|D_h l\|^2_{D_u^1\times D_u^2}) + \big([h_\gamma'(D_h u)^* D_h v]_u v, D_h p\big)_{D_u^1\times D_u^2} + 4\langle p.v, l\rangle \\ \geq \rho(\|l\|^2 + \|D_h l\|^2_{D_u^1\times D_u^2})\end{aligned} \tag{3.9}$$

holds for every $(v,l)\in U_h\times\Lambda_h$ which satisfies $e_u(u,\lambda)v + e_\lambda(u,\lambda)l = 0$.
Now we assume $(u_k,\lambda_k,p_k)\in V(u^*,\lambda^*,p^*)$. By using the formula of $\mathcal{J}_A''[\delta_u,\delta_\lambda]^2$, the last inequality and Lemma 3.2, we find that there exists constant $K_*>0$ such that

$$\mathcal{J}_A''[\delta_u,\delta_\lambda]^2 \geq \frac{K_*}{2\theta^2}\|(\delta_u,\delta_\lambda)\|^2_{U_h\times\Lambda_h} \quad \forall(\delta_u,\delta_\lambda)\in\ker(E'). \tag{3.10}$$

Therefore, (3.8) is a linear quadratic optimization problem with convex objective function. Besides, we also have that $E'(u,\lambda)$ is surjective for fixed $(u,\lambda)\in V(u^*,\lambda^*)$, hence there exists a unique solution $(\delta_u,\delta_\lambda,\delta_p,\psi)$, with $\psi\in\Lambda\big|_{A_k}$, to the following optimality system, which are stationary states of the Lagrangian $\mathcal{L}^A$

$$\begin{cases} e_u(u_k,\lambda_k)\delta_u + e_\lambda(u_k,\lambda_k)\delta_\lambda - f_1 = 0 \\ \chi_{A_k}(\delta_\lambda - f_4) = 0 \\ (2 + e_{uu}(u_k)p_k)\delta_u + e_{u\lambda}(u_k,\lambda_k)p_k\delta_\lambda + e_u(u_k,\lambda_k)\delta_p = 2f_2 \\ e_{u\lambda}(u_k,\lambda_k)p_k.\delta_u + 2\chi_{I_k}\beta(\mathbf{I}-\Delta_h)\delta_\lambda \\ \qquad\qquad + e_\lambda(u_k,\lambda_k)\delta_p + \chi_{A_k}\psi = 2\chi_{I_k}\beta(\mathbf{I}-\Delta_h)f_3. \end{cases} \tag{3.11}$$

Applying χ_{A_k} and χ_{I_k} to the last equation we get

$$\begin{cases} \chi_{I_k}\{p_k\delta_u + (u_k - f).\delta_p + \beta(\mathbf{I}-\Delta_h)\delta_\lambda\} = \chi_{I_k}\beta(\mathbf{I}-\Delta_h)f_3 \\ \chi_{A_k}\{p_k\delta_u + (u_k - f).\delta_p + \frac{1}{2}\psi\} = 0, \end{cases}$$

which implies the solvability of (3.6). We write the system (3.11) in equivalent form as

$$\begin{cases} \mathcal{J}_A''(\delta_u,\delta_\lambda) + (E')^*(\delta_p,\psi) = 2(f_2, \chi_{I_k}\beta(\mathbf{I}-\Delta_h)f_3)^T \\ E'(\delta_u,\delta_\lambda) = (f_1, \chi_A f_4). \end{cases}$$

From (2.25) it follows that $\|e_{uu}(u)[w]^*p\| \leq r\|p\|\|w\|$ for some $r>0$ independent in u and for every $p,w\in U_h$. Besides, since $0<\beta\ll 1$, we have that there exist $r_1,r_2>0$ independent in (u_k,p_k,λ_k) such that $\big\|\mathcal{J}_A''(\delta_u,\delta_\lambda)\big\|_{U_h\times U_h} \leq r_1\big[\|\delta_\lambda\| + (1+r_2)\|\delta_u\|\big]$. Therefore, from the third and the forth equations of (3.11), since $e_u(u,\lambda)$ is continuous invertible for $(u_k,p_k,\lambda_k)\in V(u^*,p^*,\lambda^*)$, there exists $r_5,r_6,K_3>0$ such that

$$\|(\delta_p,\psi)\|_{U_h\times\Lambda|_{A_k}} \leq K_3\big(\|(\delta_u, 1/(1+r_4)\delta_\lambda)\|_{U_h\times\Lambda_h} + r_5\|f_2\|_{U_h} + r_6\|f_3\|_{U_h}\big) \tag{3.12}$$

for solution $(\delta_u, \delta_\lambda, \delta_p, \psi)$ of the optimality system for auxiliary problem.
With $u \in V_1(u^*)$, $\lambda \in V(\lambda^*)$ which are mentioned in Lemma 3.2 and Assumption 3.1, E' is surjective. It follows that range$((E')^*)$ is closed and therefore the following decomposition is well-defined

$$(\delta_u, \delta_\lambda) = (\delta_u^k, \delta_\lambda^k) + (\delta_u^r, \delta_\lambda^r) \quad \text{where} \quad (\delta_u^k, \delta_\lambda^k) \in \ker(E'); (\delta_u^r, \delta_\lambda^r) \in \text{range}((E')^*).$$

From $\chi_{A_k}\beta\delta_\lambda^r = \chi_{A_k} f_4$ it follows $\|\delta_\lambda^r\| \le K\|f_4\|$ for some constant $K > 0$ which is independent to λ (since χ_{A_k} is invertible on range$((E')^*)$). Besides, from $(\delta_u^r, \delta_\lambda^r) \in \text{range}((E')^*)$ we have $e_u\delta_u^r + e_\lambda\delta_\lambda^r = 0$, and hence from Lemma 3.2, one obtains for some $K_1 > 0$ independent in λ

$$\|(\delta_u^r, \delta_\lambda^r)\|_{U_h\times\Lambda_h} \le K_1\|f_4\|. \tag{3.13}$$

Besides, since $(\delta_u^k, \delta_\lambda^k) \in \ker(E')$ and from (3.9), we have

$$\begin{aligned}
&\frac{K_*}{2\theta^2}\|(\delta_u^k, \delta_\lambda^k)\|^2_{U_h\times\Lambda_h} \le \langle \mathcal{J}''[\delta_u^k, \delta_\lambda^k], (\delta_u^k, \delta_\lambda^k)\rangle \\
=&\langle \mathcal{J}''[\delta_u, \delta_\lambda], (\delta_u, \delta_\lambda)\rangle - \langle \mathcal{J}''[\delta_u^r, \delta_\lambda^r], (\delta_u^r, \delta_\lambda^r)\rangle - 2\langle \mathcal{J}''[\delta_u^k, \delta_\lambda^k], (\delta_u^r, \delta_\lambda^r)\rangle \\
=&\langle 2(f_2, \chi_{I_k}\beta(\mathbf{I}-\Delta_h)f_3), (\delta_u, \delta_\lambda)\rangle - \langle (f_1, \chi_A f_4), (\delta_p, \psi)\rangle_{U_h\times\Lambda_h\big|_{A_k}} \\
&- \langle \mathcal{J}''[\delta_u^r, \delta_\lambda^r], (\delta_u^r, \delta_\lambda^r)\rangle - 2\langle \mathcal{J}''[\delta_u^k, \delta_\lambda^k], (\delta_u^r, \delta_\lambda^r)\rangle \\
=&\langle 2(f_2, \chi_{I_k}\beta(\mathbf{I}-\Delta_h)f_3), (\delta_u^r, \delta_\lambda^r)\rangle + \langle 2(f_2, \chi_{I_k}\beta(\mathbf{I}-\Delta_h)f_3), (\delta_u^k, \delta_\lambda^k)\rangle \\
&- \langle (f_1, \chi_A f_4), (\delta_p, \psi)\rangle_{U_h\times\Lambda_h\big|_{A_k}} - \langle \mathcal{J}''[\delta_u^r, \delta_\lambda^r], (\delta_u^r, \delta_\lambda^r)\rangle - 2\langle \mathcal{J}''[\delta_u^k, \delta_\lambda^k], (\delta_u^r, \delta_\lambda^r)\rangle.
\end{aligned} \tag{3.14}$$

From (3.12) and (3.13) it follows

$$\left|\langle (f_1, \chi_A f_4), (\delta_p, \psi)\rangle_{U_h\times\Lambda_h\big|_{A_k}}\right| \le K_4\big(\sum_{i=1}^4 \tilde r_i\|f_i\|^2\big) + \frac{K_*}{4(1+r_4)\theta^2}\|\delta_\lambda^k\|^2.$$

Besides, by the properties of e_{uu} in case $(u_k, p_k, \lambda_k) \in V(u^*, p^*, \lambda^*)$ which are mentioned in Lemma 2.1, we also have

$$\begin{aligned}
\left|2\langle \mathcal{J}''[\delta_u^k, \delta_\lambda^k], (\delta_u^r, \delta_\lambda^r)\rangle\right| &\le K_5\|f_4\|^2 + \frac{K_*}{4(1+r_4)\theta^2}\|\delta_\lambda^k\|^2; \\
\langle 2(f_2, \chi_{I_k}\beta(\mathbf{I}-\Delta_h)f_3), (\delta_u^k, \delta_\lambda^k)\rangle &\le r_7\|f_2\|^2 + r_8\|f_3\|^2 + \frac{K_* r_4}{4(1+r_4)\theta^2}\|(\delta_u^k, \delta_\lambda^k)\|^2_{U_h\times\Lambda_h}; \\
-\langle \mathcal{J}''[\delta_u^r, \delta_\lambda^r], (\delta_u^r, \delta_\lambda^r)\rangle &\le -\langle p_k, e_{uu}(u_k)[\delta_u^r]^2\rangle - 4\langle p_k\delta_u^r, \delta_\lambda^r\rangle \\
&\le r_9\|\delta_u^r\|^2 + r_{10}\|\delta_\lambda^r\|^2 \le r_{11}\|(\delta_u^r, \delta_\lambda^r)\|^2_{U_h\times\Lambda_h} \le r_{12}\|f_4\|^2.
\end{aligned}$$

where $\tilde r_i, r_i$ and K_i stand for positive constants. These relations imply that for some constants $k_1, k_2, k_3, k_4 > 0$

$$\frac{r_4 K_*}{4(1+r_4)\theta^2}\|(\delta_u^k, \delta_\lambda^k)\|^2_{U_h\times\Lambda_h} \le k_1\|f_1\|^2 + k_2\|f_2\|^2 + k_3\|f_3\|^2 + k_4\|f_4\|^2.$$

By combining the last relation with (3.12), we find that there exists $\kappa > 0$, which are independent in (u_k, p_k, λ_k) such that

$$\|(\delta_u, \delta_\lambda, \delta_p, \psi)\|^2_{U_h\times\Lambda_h\times U_h\times\Lambda_h\big|_{A_k}} \le \kappa\|(f_1, f_2, f_3, f_4)\|^2_{U_h\times U_h\times\Lambda_h\times U_h}$$

and therefore, we have for some $\bar\kappa > 0$

$$\|\delta\|^2_{H_h} \le \bar\kappa\|(F_h^1, F_h^2, F_h^3, F_h^4, F_h^5)\|^2_{H_h'}.$$

The last relation implies $\mathcal{G}_{F_h}(\mathbf{y})$ is uniformly continuously invertible in $V(\mathbf{y}^*)$ and futhermore, it follows that we have $\|\mathcal{G}_{F_h}(\mathbf{y})^{-1}\|_{\mathcal{L}(H_h, H_h')} \le \bar\kappa$. ☐

3.4. **Globalization.** The standard semismooth Newton method (3.4) typically exhibits a very small convergence neighbourhood for high values of γ. In order to globalize the semismooth Newton method, we consider a modified Jacobi matrix in each iteration. The main idea consists in reinforcing feasibility of the dual quantities (with suitable projections) in the building of the Jacobian.

To describe the modification, let us first introduce the following notation:

$$P_1(u) = \frac{2\gamma - 1}{4\gamma} + \frac{\gamma |D_h u|}{2} - \frac{\gamma}{2} t_1(u) t_2(u) + \frac{\gamma^3}{2} t_1^2(u) t_2^2(u);$$
$$P_2(u) = \frac{\gamma}{2} - \frac{\gamma^2}{2}\big[t_1(u) + t_2(u)\big] + \gamma^3 \big[t_1(u) + t_2(u)\big] t_1(u) t_2(u).$$

The proposed building process is based on the proved properties of the stationary point we look for. Indeed, at the solution $\mathbf{y}^*$, we have the following:

- On $\mathcal{A}_\gamma$: $q = h_\gamma(D_h u^*) = \frac{D_h u^*}{|D_h u^*|}$. On the other hand, $h'_\gamma(D_h u)^* D_h z = \frac{D_h z}{|D_h u|} - \frac{\langle D_h u, D_h z\rangle}{|D_h u|^2} \frac{D_h u}{|D_h u|}$. Since $\left|\frac{D_h u^*}{|D_h u^*|}\right| \le 1$, by projecting onto feasible set, we have an approximation of $h'_\gamma(D_h u) D_h$ on $\mathcal{A}_\gamma$:
$$(h'_\gamma(D_h u))^\dagger D_h = \frac{D_h}{|D_h u|} - \frac{\langle D_h u, D_h\rangle}{|D_h u|^2} \frac{q}{\max\{1, |q|\}}.$$
- On $\mathcal{S}_\gamma$: $q = h_\gamma(D_h u^*) = P_1(u) \frac{D_h u^*}{|D_h u^*|}$, $1 - \frac{1}{2\gamma} \le P_1(u) \le 1$ and
$$\begin{aligned} h'_\gamma(D_h u)^* D_h z &= P_1(u)\left(\frac{D_h z}{|D_h u|} - \frac{\langle D_h u, D_h z\rangle}{|D_h u|^3} D_h u\right) + P_2(u) \frac{\langle D_h u, D_h z\rangle}{|D_h u|^2} D_h u \\ &= \left(\frac{(D_h z D u^T)}{|D_h u|^2} - \frac{\langle D_h u, D_h z\rangle}{|D_h u|^2}\right) P_1(u) \frac{D_h u}{|D_h u|} + P_2(u) \frac{\langle D_h u, D_h z\rangle}{|D_h u|^2} D_h u. \end{aligned}$$
Hence, similar to the above consideration, we have approximation of $h'_\gamma(D_h u) D_h$ on $\mathcal{S}_\gamma$:
$$(h'_\gamma(D_h u))^\dagger D_h = \left\{\frac{(D_h z D u^T)}{|D_h u|^2} + \left[\frac{P_2(u)}{P_1(u)} - \frac{1}{|D_h u|}\right] \frac{\langle D_h u, D_h\rangle}{|D_h u|}\right\} \frac{q}{\max\{1, |q|\}}.$$

By replacing $(h'_\gamma(D_h u))$ by $(h'_\gamma(D_h u))^\dagger$, we get a modified generalized derivative of F_h:

$$\mathcal{J}^\dagger_{F_h}(\mathbf{y})(\delta_u, \delta_q, \delta_p, \delta_z, \delta_\lambda)^T = \tag{3.15}$$
$$\begin{pmatrix} (2\lambda \mathbf{I} - \mu \Delta_h)\delta_u - \mathrm{Div}_h \delta_q + 2(u - f).\delta_\lambda \\ (h'_\gamma(D_h u))^\dagger \delta_u - \delta_q \\ 2\delta_u + (2\lambda \mathbf{I} - \mu\Delta_h)\delta_p - \mathrm{Div}_h \delta_z + 2p.\delta_\lambda \\ \big(h'_\gamma(D_h u)^* D_h p\big)_u \delta_u + (h'_\gamma(D_h u))^\dagger \delta_p - \delta_z \\ p.\delta_u + (u - f).\delta_p + \beta(\mathbf{I} - \Delta_h)\delta_\lambda - G_m\big((u - f).p - \beta \Delta_h \lambda\big)\big(p.\delta_u + (u - f).\delta_p - \beta \Delta_h \delta_\lambda\big) \end{pmatrix}$$

and the corresponding modified iteration for solving of $F_h(\mathbf{y}) = 0$ with F_h in (3.1):

$$\mathcal{J}^\dagger_{F_h}(\tilde{\mathbf{y}}_k)\big(\tilde{\mathbf{y}}_{k+1} - \tilde{\mathbf{y}}_k\big) = -F_h(\tilde{\mathbf{y}}_k). \tag{3.16}$$

4. Computational experiments

All methods and schemes developed previously were implemented in MATLAB and run in a HP Blade multiprocessor system. The overall used algorithm is given though the following steps:

Algorithm 4.1 (Domain decomposition-semismooth Newton algorithm)**.**

0. *Initialize* $\mathbf{y}_0 = (u_0, q_0, p_0, z_0, \lambda_0)^T$ *and set* $k = 0$.

1. *At step* $k \geq 0$*, with* $\mathbf{y}_k$ *is known, solve* $\delta_{\mathbf{y}} = (\delta_u, \delta_q, \delta_p, \delta_z, \delta_\lambda)^T$ *from* (3.4)
$$\mathcal{G}_{F_h}(\mathbf{y}_k)\delta_{\mathbf{y}} = -F_h(\mathbf{y}_k)$$
end update $\mathbf{y}_{k+1} = \mathbf{y}_k + \delta_{\mathbf{y}}$*.*
2. *Check stopping condition. If is not satisfied, set* $k = k+1$ *and repeat step 1.*

Since the computations in each subdomain are independent from each other, these may run simultaneously in parallel processors. We implemented a standard for-loop for iteration k of the domain decomposition method and, within each k, a parallel MATLAB parfor-loop with index n for computing the solution on each subdomain.

For the numerical experimentation we introduce some notation and several quantities of interest, which are described next:

L	Number of overlapping pixels
M_{NonDDC}	Semismooth Newton method on the whole domain Ω
M_{orgDDC}	Semismooth Newton method with original Schwarz method
M_{optDDC}	Semismooth Newton method with optimized Schwarz method
er_λ	$\|\lambda_{DD} - \lambda\|$, where λ_{DD}) is obtained by M_{orgDDC} or M_{optDDC}, and λ by M_{NonDDC}
er_u	$\|u_{DD} - u\|$, where u_{DD} is obtained by M_{orgDDC} or M_{optDDC}, and u by M_{NonDDC}.
$k_{\max}$	Maximum number of subdomain SSN-iterations in all DD iterations
SSNR	$\sum_i \|F_h^i(\mathbf{y}_{k_{\max}})\|$ on $\Omega_i \subset \Omega$
T_p	Performing time (in seconds).

We also use the recently proposed structural similarity measure (MSSIM) (see [30]) to compare the obtained images with the original one.

4.1. **Uniform noise.** In this first experiment, we consider the denoising problem with brain scan images. The first set consists of images of 256×256 pixels and Gaussian noise with zero mean and variance $\sigma = 0.0075$. The original and noisy images are shown in Figure 4.1. The domain decomposition-semismooth Newton algorithms run with the parameter values $\gamma = 50$, $\mu = 10^{-13}$, $\beta = 10^{-9}$, $h = 0.01$ and the stopping criteria $SSNR \leq E_r = 10^{-5}$. The results are shown in Figure 4.2. From the surface representation of λ, we can observe that λ is continuous and its shape is related to the one of the original image.

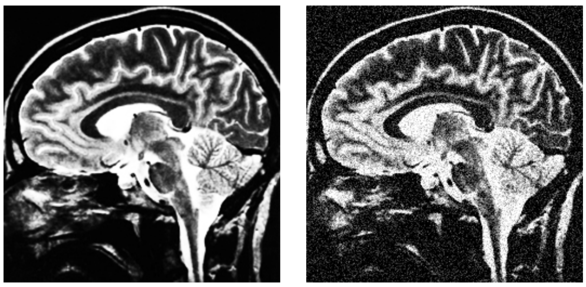


FIGURE 4.1. The first experiment: Original (left) and noisy (right) images.

In Table 4.1 the performance of the different methods is compared. For all of them, only the first 2 domain decomposition iterations were considered. The total number of SSN iterations differ at most by one. The impact of the domain decomposition method becomes clear when comparing the computing times of the methods, corresponding to one, two and four subdomains. The computing time is significantly reduced. The effect of the optimized transmission conditions can be realized when comparing the gap between subdomains, which is much lower in the case of optimized transmission conditions (M_{optDDC}) than in the standard Schwarz method (M_{orgDDC}).

4.2. **Non-uniform noise.** For this experiment we consider input images of size 512×512, with a Gaussian noise of $\sigma = 0.014$ on the whole domain and an additional noise of 0.016 on some areas which are marked in red (see Figure 4.3). The parameter values used are $\mu = 0$,

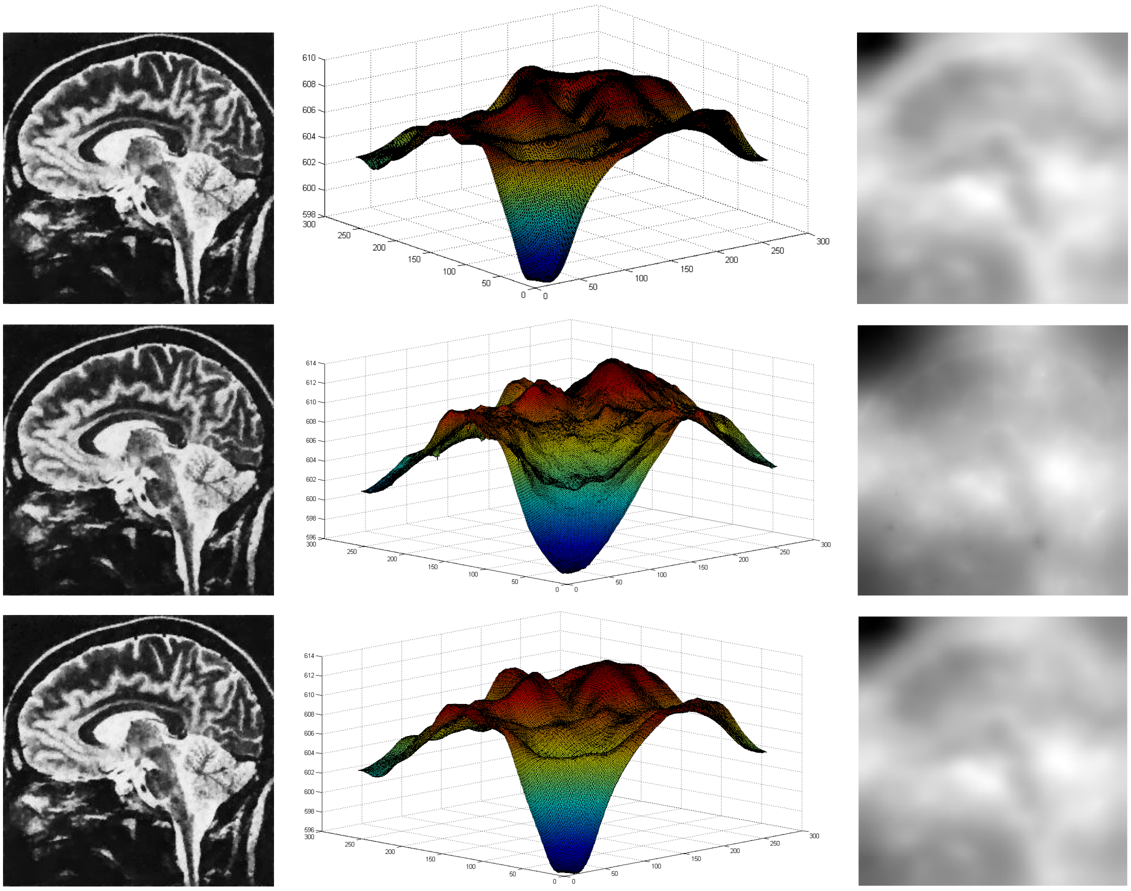


FIGURE 4.2. Using the training set in Figure 4.1 the optimally denoised images are shown (left), surface plots of λ (center) and images of λ (right). The first row corresponds to the result achieved without domain decomposition M_{NonDDC}; the second and third row correspond to the results using domain decomposition (2 iterations) without (M_{orgDDC}) and with (M_{optDDC}) optimized transmission condition, respectively. Here we used 2 subdomains with an overlap of $L = 40$ pixels.

Method	$k_{\max}$	$L=20$				$L=40$				$L=75$			
		(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
M_{NonDDC}	10	$MSSIM = 0.894 \quad T_p = 83.71$											
M_{orgDDC}	(a) 11	0.851	5.3	2.71	28.11	0.861	3.1	1.76	38.01				
	(b) 11	0.853	5.9	3.60	10.09	0.858	3.7	2.05	19.99				
M_{optDDC}	(a) 11	0.869	3.2	0.99	29.85	0.881	1.9	1.01	39.92	0.883	1.5	0.82	47.03
	(b) 10	0.865	3.6	1.22	11.03	0.877	2.3	1.09	23.81				

TABLE 4.1. Numerical results for the first experiment. Rows (a): 2 subdomains; (b): 4 subdomains. Columns (1): $MSSIM$; (2): er_u ($\times 10^{-3}$); (3): er_λ; (4): T_p.

$\beta = 10^{-10}$, $\gamma = 100$ and $h = 0.002$ and the stopping criteria is $SSNR \leq E_r = 10^{-4}$. The shape of λ is shown in Figure 4.4.

The semismooth Newton method, on the whole domain, takes $k_{\max} = 14$ iterations and $T_p = 1398.1(s)$ to converge. The denoised image has an $MSSIM = 0.791$. Meanwhile, the M_{orgDDC} with $L = 30$ takes $k_{\max} = 15$ iterations and $T_p = 411.7(s)$ to converge, and yields $MSSIM = 0.769$. The error with respect to λ is given by $er_\lambda = 0.97$. With the same value $L = 30$, the M_{optDDC} stops after $k_{\max} = 15$ and $T_p = 433.9(s)$. The similarity measure is $MSSIM = 0.785$ and the error with respect to λ is given by $er_\lambda = 0.51$. The corresponding images for all three methods are given in Figures 4.4, 4.5 and 4.6, respectively.

From Figures 4.4, 4.5 and 4.6 we can observe that the areas with higher noisy level result in smaller pointwise values of λ. Moreover, from the tabulated results, one can realize that,

FIGURE 4.3. The input images for the non-uniform noise experiment: original (left) and noisy (right) images.

FIGURE 4.4. First row: Denoised image (left) and image form of λ (right).

FIGURE 4.5. M_{orgDDC} with $L = 30$: Denoised image (left) and λ (right).

FIGURE 4.6. M_{optDDC} with $L = 30$: Denoised image (left) and λ (right).

in order to get good results for M_{orgDCC}, a sufficiently large value of L is required. This has of course an increasing effect in the total computing time.

4.3. **Large training set.** In the next experiment, we compute the optimal functional parameter λ from a training set of 10 pairs $(u_j^\dagger, f_j)$, $j = 1, \ldots, 10$. The images (of size 256×256)

were taken from the OASIS online database. A Gaussian noise with $\sigma = 0.025$ was distributed on the images, and in the areas marked by red, additional noise with $\sigma = 0.1$ was imposed (to all noised images at the same location).

The parameter values for this experiment are $\gamma = 50$, $\mu = 10^{-15}$, $\beta = 10^{-12}$ and $h = 1/256$. We utilized the optimized Schwarz method M_{optDDC}, with overlapping size $L = 5$, and stop after two iterates. A total amount of 24 subdomains were considered and the computations were carried out in parallel. The semismooth Newton method, within each step of M_{optDDC}, stops whenever $err < 0.01$. The results are shown in Figure 4.7.

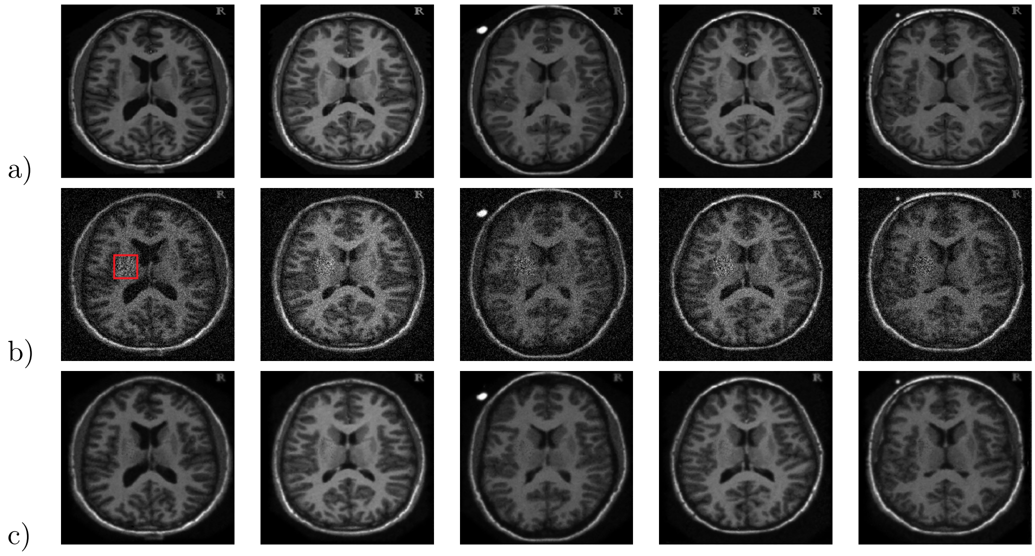


FIGURE 4.7. Results of learning a spatial parameter λ for a training set $(u_k^\dagger, f_k)$: (a) Original image, (b) Noisy image, (c) Denoised image with M_{optDDC} (24 subdomains).

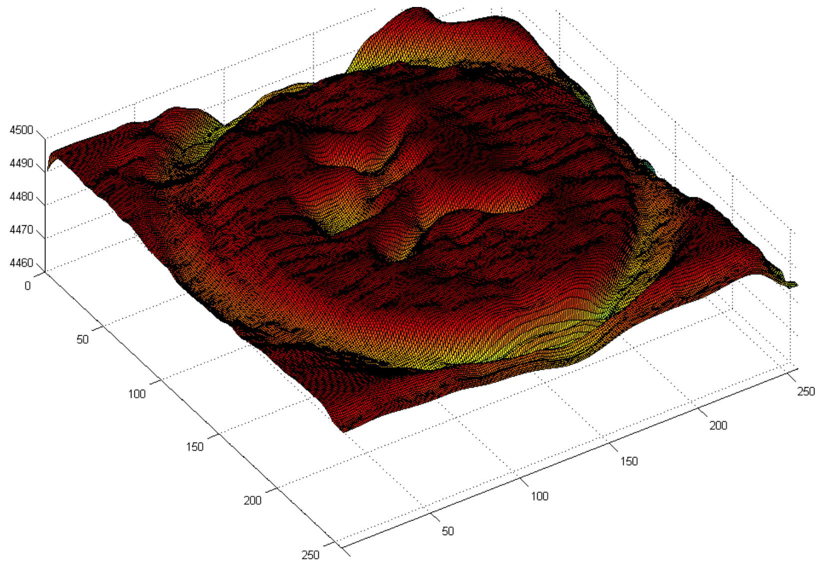


FIGURE 4.8. Optimal parameter λ for the experiment in Figure 4.7 after 2 Schwarz iterations.

The performance of the overall algorithm for the cases of 4 and 24 subdomains is registered in Table 4.2. It becomes clear from the data, that there is a significant decrease in the total computing time, when an increasing number of subdomains is considered. This, on the other hand, does not significantly affect the quality of the obtained image, measured by MSSIM.

We denote $AVG_{Gap_\lambda} := \frac{1}{10}\sum_{i=j}^{10} \|\lambda_j^m - \lambda_j^n\|\big|_{\Omega_m\cap\Omega_n}$, $j = 1,\ldots,10$, $\lambda_j^l = \lambda_j\big|_{\Omega_l}$ and Ω_m, Ω_n are subdomains.

$\#\Omega_i$	$k_{\max}$	T_p	$MSSIM_{\min}$	$MSSIM_{\max}$	$MSSIM_{avg}$	AVG_{Gap_λ}
4	17	2098.42	0.826	0.878	0.856	3.072
24	14	179.01	0.821	0.883	0.863	2.785

TABLE 4.2. Numerical results for M_{optDDC}. $MSSIM_{\min}$, $MSSIM_{\max}$, $MSSIM_{avg}$: min, max and average MSSIM for u_j^k with respect to $u_j^\dagger$, $j = 1..10$.

4.4. **Performance compared to other spatially-dependent approaches.** In the last experiment, we compare the results of our approach with the ones obtained with the spatially adapted total variation method (SA-TV) proposed in [8]. For the comparison, we use the well-known "cameraman" image (of size 256×256). The degraded image is corrupted with a Gaussian noise with zero mean and variance $\sigma = 0.01$.

On a grid of size $h = 0.01$, the chosen parameters for M_{NonDDC} are $\gamma = 100$, $\mu = 10^{-14}$, $\beta = 10^{-13}$ and that for SA-TV are $\bar{\mu} = 0$ and $\bar{\beta} = 10^{-3}$. We use the stopping rule as in [8], i.e., $\|u_k - f\| \le \sigma$.

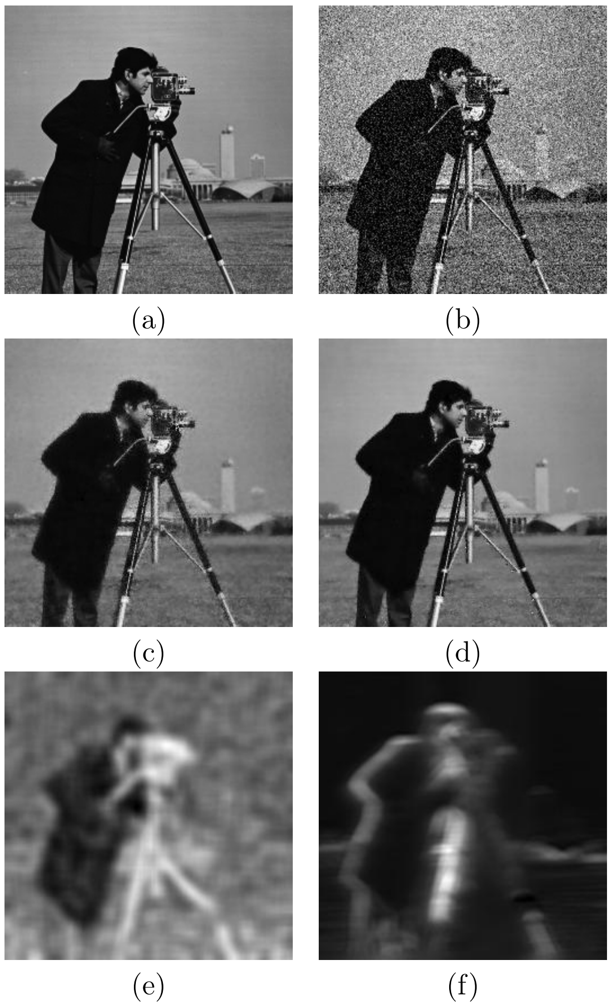


FIGURE 4.9. Comparison of SA-TV and M_{NonDDC}: (a) Original image, (b) Noise image, (c) Denoising image of SA-TV, (d) Denoising image of M_{NonDDC}, (e) $\bar{\lambda}$ of SA-TV, (f) λ of M_{NonDDC}.

The performance of SA-TV and M_{NonDDC} is compared quantitatively by means of the peak signal-to-noise ratio (PSNR) and MSSIM. The results of two methods are shown in

Table 4.3, where it can be observed that our approach outperforms the SA-TV for the tested image, with respect to both quality measures: (PSNR) and (MSSIM). Of course, in our approach there is an additional source of information through the training pair.

Concerning the total computing time, the M_{NonDDC} requires less seconds, even without domain decomposition. As we discussed in the previous experiments, the computing time can be significantly reduced by utilizing M_{orgDCC} or M_{optDCC}.

Method	$k_{\max}$	T_p	PSNR	MSSIM
SA-TV	9	35.67	27.71	0.807
M_{NonDDC}	4	24.01	28.35	0.843

TABLE 4.3. Comparison of SA-TV and M_{NonDDC}

5. Appendix

Proof of Lemma 2.1. For $z, \xi, \tau \in \mathbb{R}^2$, by setting $t_1^z = \frac{\gamma}{2}\big(\gamma|z| - 1 + \frac{1}{2\gamma}\big)$, $t_2^z = \gamma|z| - 1 - \frac{3}{2\gamma}$,

$$\chi_{\mathcal{A}_z} = \begin{cases} 1 & \text{if} \quad \gamma|z| \geq b \\ 0 & \text{otherwise} \end{cases} ; \chi_{\mathcal{S}_z} = \begin{cases} 1 & \text{if} \quad a \leq \gamma|z| < b \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \chi_{\mathcal{I}_z} = \begin{cases} 1 & \text{if} \quad \gamma|z| < a \\ 0 & \text{otherwise} \end{cases}$$

we get
$$h'_\gamma(z)[\xi] = \chi_{\mathcal{A}_z}\left[\frac{\xi}{|z|} - \frac{\langle z,\xi\rangle}{|z|^3}z\right] + \chi_{\mathcal{S}_z}\left\{\frac{\gamma}{2}\xi + \gamma^2\big(\gamma|z|-1\big)\big(2\gamma^2 t_1^z t_2^z - 1\big)\frac{\langle z,\xi\rangle}{|z|^2}z \right.$$
$$\left. + \left[\frac{2\gamma-1}{4\gamma} - \frac{\gamma t_1^z t_2^z}{2} + \frac{\gamma^3 (t_1^z t_2^z)^2}{2}\right]\left(\frac{\xi}{|z|} - \frac{\langle z,\xi\rangle}{|z|^3}z\right)\right\} + \chi_{\mathcal{I}_z}\big(\gamma\xi\big).$$

Moreover, by setting $\phi(z,\xi) = -\left\{\frac{(\xi z^T)}{|z|^3} + \frac{(z\xi^T)}{|z|^3} - 3\frac{\langle z,\xi\rangle(zz^T)}{|z|^5} + \frac{\langle z,\xi\rangle}{|z|^3}\right\}$, we get

$$h''_\gamma(z)[\xi,\tau] = \chi_{\mathcal{A}_z}\phi(z,\xi)\tau + \chi_{\mathcal{S}_z}\left\{\phi(z,\xi)\tau\left[\frac{\gamma}{2}t_1^z t_2^z\Big(4\gamma^3|z|\big(\gamma|z|-1\big) - \gamma^2 t_1^z t_2^z + 1\Big)\right.\right.$$
$$\left.\left. -\Big(\gamma^3|z|^2 - \gamma^2|z| + \frac{1}{2} - \frac{1}{4\gamma}\Big)\right] + 6\gamma^5 t_1^z t_2^z \frac{\langle z,\xi\rangle(zz^T)}{|z|^3}\tau\right\}.$$

a) We first consider the case $z, \hat{z}, \xi, \tau \in \mathbb{R}^2$. Indeed,
($a1$) If $|z| < \frac{a}{\gamma}$ and $|\hat{z}| < \frac{a}{\gamma}$, we have $|R(z,\hat{z},\xi,\tau)| = |h''_\gamma(z)[\xi][\tau] - h''_\gamma(\hat{z})[\xi][\tau]| = 0$.
($a2$) If $|z| > \frac{b}{\gamma}$ and $|\hat{z}| < \frac{a}{\gamma}$, by a straight computation, we find $|z - \hat{z}| \geq \big||z| - |\hat{z}|\big| \geq \frac{1}{2\gamma^2}$ and $\big|R(z,\hat{z},\xi,\tau)\big| = \big|\phi(z,\xi)\tau\big| \leq \frac{24\gamma^4}{(2\gamma+1)^2}|\xi||\tau|$. This yields (2.3).
($a3$) If $|z|, |\hat{z}| > \frac{b}{\gamma}$, we have $\frac{1}{|\hat{z}|^3}, \frac{1}{|z|^3} \leq \big(\frac{1}{\gamma} + \frac{1}{2\gamma^2}\big)^{-3}$ and

$$R(z,\hat{z},\xi,\tau) = \left\{3\left[\frac{\langle z,\xi\rangle\big(zz^T\big)}{|z|^5} - \frac{\langle \hat{z},\xi\rangle\big(\hat{z}\hat{z}^T\big)}{|\hat{z}|^5}\right] - \left[\frac{\big(\xi z^T\big)}{|z|^3} - \frac{\big(\xi\hat{z}^T\big)}{|\hat{z}|^3}\right] - \left[\frac{\big(z\xi^T\big)}{|z|^3} - \frac{\big(\hat{z}\xi^T\big)}{|\hat{z}|^3}\right]\right.$$
$$\left. - \left[\frac{\langle z,\xi\rangle}{|z|^3} - \frac{\langle\hat{z},\xi\rangle}{|\hat{z}|^3}\right]\right\}\tau =: (3S_0 - S_1 - S_2 - S_3)\tau.$$

One gets $|S_1| = \big|\frac{\langle z,\xi\rangle}{|z|^3} - \frac{\langle\hat{z},\xi\rangle}{|\hat{z}|^3}\big| \leq \big[\frac{1}{|\hat{z}|^3} + \frac{1}{|z|^3}\big]\big|\langle z-\hat{z},\xi\rangle\big| + \frac{\big||z|^3\langle z,\xi\rangle - |\hat{z}|^3\langle\hat{z},\xi\rangle\big|}{|z|^3|\hat{z}|^3}$. We find for the first term $\big[\frac{1}{|\hat{z}|^3} + \frac{1}{|z|^3}\big]\big|\langle z-\hat{z},\xi\rangle\big| \leq \frac{16\gamma^6}{(2\gamma+1)^3}|z-\hat{z}||\xi|$ and for the second

$$\begin{aligned}
\frac{\big||z|^3\langle z,\xi\rangle - |\hat{z}|^3\langle\hat{z},\xi\rangle\big|}{|z|^3|\hat{z}|^3} &\leq \frac{|\xi|}{|z|^3|\hat{z}|^3}\big||z|^3 z - |\hat{z}|^3\hat{z}\big| \\
&= \frac{|\xi|}{|z|^3|\hat{z}|^3}\big||\hat{z}|^3(z-\hat{z}) + z\big[|z|^3 - |\hat{z}|^3\big]\big| \\
&\leq |\xi|.|z-\hat{z}|\Big[\frac{1}{|z|^3} + \frac{1}{|\hat{z}|^3} + \frac{1}{|z|.|\hat{z}|^2} + \frac{1}{|z|^2|\hat{z}|}\Big] \leq \frac{32\gamma^6}{(2\gamma+1)^3}|z-\hat{z}||\xi|.
\end{aligned}$$

Hence, $|S_1\tau| \le \frac{48\gamma^6}{(2\gamma+1)^3}|z-\hat{z}||\xi||\tau|$.

We also have $|S_2\tau| = \frac{\left|(|\hat{z}|^3 z - |z|^3\hat{z})\langle \xi,\tau\rangle\right|}{|z|^3|\hat{z}|^3} \le \left[\frac{1}{|\hat{z}|^3}+\frac{1}{|z|^3}\right]\left|z-\hat{z}\right||\xi||\tau| + \frac{\left||z|^3 z - |\hat{z}|^3\hat{z}\right|}{|z|^3|\hat{z}|^3}|\xi||\tau|$ and $|S_3\tau| = \left|\frac{\xi\langle z,\tau\rangle}{|z|^3} - \frac{\xi\langle \hat{z},\tau\rangle}{|\hat{z}|^3}\right| = \left|\xi\langle \frac{|\hat{z}|^3 z - |z|^3\hat{z}}{|z|^3|\hat{z}|^3},\tau\rangle\right| \le |\xi|\left|\frac{|\hat{z}|^3 z - |z|^3\hat{z}}{|z|^3|\hat{z}|^3}\right||\tau|$.

Similarly, we have $|S_2\tau| \le \frac{48\gamma^6}{(2\gamma+1)^3}|z-\hat{z}||\xi||\tau|, |S_3\tau| \le \frac{32\gamma^6}{(2\gamma+1)^3}|z-\hat{z}||\xi||\tau|$.

We get $|S_0\tau| \le |\xi||\tau|\left[\left|\frac{|\hat{z}|^3 z - |z|^3\hat{z}}{|z|^3|\hat{z}|^3}\right| + \left|\frac{(z_1z_2)z}{|z|^5} - \frac{(\hat{z}_1\hat{z}_2)\hat{z}}{|\hat{z}|^5}\right|\right]$, where $z=(z_1,z_2), \hat{z}=(\hat{z}_1,\hat{z}_2)$. Similar to $S_3\tau$, we have $\left|\frac{|\hat{z}|^3 z - |z|^3\hat{z}}{|z|^3|\hat{z}|^3}\right| \le \frac{32\gamma^6}{(2\gamma+1)^3}|z-\hat{z}|$. By setting $\bar{z}=(\bar{z}_1,\bar{z}_2)=\frac{z}{|z|}$ and $\tilde{z}=(\tilde{z}_1,\tilde{z}_2)=\frac{\hat{z}}{|\hat{z}|}$ one gets $\left|\frac{(z_1z_2)z}{|z|^5} - \frac{(\hat{z}_1\hat{z}_2)\hat{z}}{|\hat{z}|^5}\right| \le \left[\frac{1}{|\hat{z}|^3}+\frac{1}{|z|^3}\right]|z-\hat{z}| + \left|\frac{|z|^3(\tilde{z}_1\tilde{z}_2)z - |\hat{z}|^3(\bar{z}_1\bar{z}_2)\hat{z}}{|z|^3|\hat{z}|^3}\right|$.

We find $\left|\frac{|z|^3(\tilde{z}_1\tilde{z}_2)z - |\hat{z}|^3(\bar{z}_1\bar{z}_2)\hat{z}}{|z|^3|\hat{z}|^3}\right| \le \frac{|(\tilde{z}_1\tilde{z}_2)z - (\bar{z}_1\bar{z}_2)\hat{z}|}{|\hat{z}|^3} + |z-\hat{z}|\left[\frac{1}{|z||\hat{z}|^2}+\frac{1}{|z|^2|\hat{z}|}+\frac{1}{|z|^3}\right]$. Without loss of generality, we assume that $|z|\le|\hat{z}|$. One can verify that $\left|(\tilde{z}_1\tilde{z}_2)z - (\bar{z}_1\bar{z}_2)\hat{z}\right| \le \frac{|z-\hat{z}|}{2} + \frac{|\hat{z}|}{2}|\tilde{z}-\bar{z}||\tilde{z}+\bar{z}|$ and $|\tilde{z}-\bar{z}| \le \frac{2|\hat{z}-z|}{|z|}$. It follows $\frac{\left|(\tilde{z}_1\tilde{z}_2)z - (\bar{z}_1\bar{z}_2)\hat{z}\right|}{|\hat{z}|^3} \le \frac{5|z-\hat{z}|}{2|\hat{z}|^3}$. Hence, we have $|S_0\tau| \le |\xi||\tau||z-\hat{z}|\left\{\frac{32\gamma^6}{(2\gamma+1)^3}+\frac{2}{|z|^3}+\frac{7}{2|\hat{z}|^3}+\frac{1}{|z||\hat{z}|^2}+\frac{1}{|z|^2|\hat{z}|}\right\} \le \frac{96\gamma^6}{(2\gamma+1)^3}|z-\hat{z}||\xi||\tau|$ and therefore, $|R(z,\hat{z},\xi,\tau)| \le \frac{220\gamma^6}{(2\gamma+1)^3}|z-\hat{z}||\xi||\tau|$.

$(a4)$ If $a\le\gamma|z|,\gamma|\hat{z}|\le b$ then $0\le t_1^z, t_1^{\hat{z}} \le \frac{1}{\gamma}$; $-\frac{1}{\gamma}\le t_2^z, t_2^{\hat{z}}\le 0$ and $|\phi(z,\xi)|,|\phi(\hat{z},\xi)| \le \frac{24\gamma^4|\xi|}{(2\gamma-1)^2}$. By setting $q(z)=\frac{\gamma}{2}t_1^zt_2^z\left[4\gamma^3|z|\big(\gamma|z|-1\big)-\gamma^2t_1^zt_2^z+1\right]-\left[\gamma^3|z|^2-\gamma^2|z|+\frac{1}{2}-\frac{1}{4\gamma}\right]$ we have

$$R(z,\hat{z},\xi,\tau)=\left\{\left[q(z)\phi(z,\xi)-q(\hat{z})\phi(\hat{z},\xi)\right]+6\gamma^5\left[\frac{t_1^zt_2^z\langle z,\xi\rangle(zz^T)}{|z|^3}-\frac{t_1^{\hat{z}}t_2^{\hat{z}}\langle \hat{z},\xi\rangle(\hat{z}\hat{z}^T)}{|\hat{z}|^3}\right]\right\}\tau$$

and $|q(z)|,|q(\hat{z})| \le \gamma(1+\frac{1}{2\gamma})(2+\frac{1}{2\gamma})+\frac{6\gamma+5}{4\gamma}$. We now analyze each term.

$$\left|\left[q(z)\phi(z,\xi)-q(\hat{z})\phi(\hat{z},\xi)\right]\tau\right| \le \left|q(z)-q(\hat{z})\right|\left|\phi(z,\xi)\right||\tau| + |q(\hat{z})|\left|\phi(z,\xi)-\phi(\hat{z},\xi)\right||\tau|.$$

Similarly for (a3), we get $\left|\left[\phi(z,\xi)-\phi(\hat{z},\xi)\right]\tau\right| \le \frac{220\gamma^6}{(2\gamma-1)^3}|z-\hat{z}||\xi||\tau|$. Besides,

$$\left|q(z)-q(\hat{z})\right| \le \frac{\gamma}{2}\left|t_1^zt_2^z-t_1^{\hat{z}}t_2^{\hat{z}}\right|\left|4\gamma^3|z|\big(\gamma|z|-1\big)-\gamma^2t_1^zt_2^z+1\right|$$

$$+\frac{\gamma}{2}\left|t_1^{\hat{z}}t_2^{\hat{z}}\right|\left[4\gamma^4\left||z|^2-|\hat{z}|^2\right|+\gamma^3\left||z|-|\hat{z}|\right|+\gamma^2\left|t_1^zt_2^z-t_1^{\hat{z}}t_2^{\hat{z}}\right|\right]+\gamma^3\left||z|^2-|\hat{z}|^2\right|+\gamma^2\left||z|-|\hat{z}|\right|.$$

From $t_1^zt_2^z=\gamma^2|z|^2-(a+b)|z|+ab$, it follows $\left|t_1^zt_2^z-t_1^{\hat{z}}t_2^{\hat{z}}\right| \le \gamma^2\left||z|^2-|\hat{z}|^2\right|+|a+b|\left||z|-|\hat{z}|\right|$. Note that $\left||z|^2-|\hat{z}|^2\right| = \left|(|z|-|\hat{z}|)(|z|+|\hat{z}|)\right| \le \frac{2\gamma+1}{\gamma^2}|z-\hat{z}|$. Hence, there exists constant $m_1(\gamma)>0$ only dependent on γ, such that $\left|\left[q(z)\phi(z,\xi)-q(\hat{z})\phi(\hat{z},\xi)\right]\tau\right| \le m_1(\gamma)|z-\hat{z}||\xi||\tau|$. For the second term $\left|\frac{t_1^zt_2^z\langle z,\xi\rangle(zz^T)}{|z|^3}-\frac{t_1^{\hat{z}}t_2^{\hat{z}}\langle \hat{z},\xi\rangle(\hat{z}\hat{z}^T)}{|\hat{z}|^3}\right| =: T_2(z,\hat{z},\xi)$, we have

$$T_2(z,\hat{z},\xi) \le \frac{|t_1^zt_2^z-t_1^{\hat{z}}t_2^{\hat{z}}|\left|\langle z,\xi\rangle(zz^T)\right|}{|z|^3} + |t_1^{\hat{z}}t_2^{\hat{z}}|\left|\frac{\langle z,\xi\rangle(zz^T)}{|z|^3}-\frac{\langle \hat{z},\xi\rangle(\hat{z}\hat{z}^T)}{|\hat{z}|^3}\right|.$$

We get again the expressions as in the first term and case $(a3)$. Hence, there exists a constant $m_2(\gamma)>0$ only depending in γ, such that $|R(z,\hat{z},\xi,\tau)| \le m_2(\gamma)|z-\hat{z}||\xi||\tau|$.

$(a5)$ If $a\le\gamma|z|\le b$ and $\gamma|\hat{z}|<a$ then $h''(\hat{z})[\xi][\tau]=0$ and hence $|R(z,\hat{z},\xi,\tau)| = |h''(z)[\xi][\tau]|$. Similarly to cases (a3) and (a4), we have $|\phi(z,\xi)||\tau| \le \frac{24\gamma^4|\xi||\tau|}{(2\gamma-1)^2}$ and $\frac{\left|\langle z,\xi\rangle(zz^T)\tau\right|}{|z|^3} \le |\xi||\tau|$. From $|t_1^z|,|t_2^z| \le \frac{1}{\gamma}$ it follows that $\frac{\gamma}{2}|t_1^zt_2^z|\left|4\gamma^3|z|\big(\gamma|z|-1\big)-\gamma^2t_1^zt_2^z+1\right| \le (\gamma+\frac{3}{2})|t_1^z|$ and $6\gamma^5|t_1^zt_2^z|\left|\frac{\langle z,\xi\rangle(zz^T)}{|z|^3}\right| \le 6\gamma^4|t_1^z||\xi|$.

Note that $0\le\gamma|\hat{z}|\le a\le\gamma|z|$, hence $0\le t_1^z=\gamma|z|-a\le\gamma|z|-\gamma|\hat{z}|$ and therefore $|t_1^z|\le\gamma(|z|-|\hat{z}|)\le\gamma|z-\hat{z}|$. Besides, $\left|\gamma^3|z|^2-\gamma^2|z|+\frac{1}{2}-\frac{1}{4\gamma}\right| = \gamma\left|(\gamma|z|-\frac{1}{2\gamma})(\gamma|z|-1+\frac{1}{2\gamma})\right| = \gamma\left|\gamma|z|-\frac{1}{2\gamma}\right||t_1^z| \le \gamma^2|z-\hat{z}|$. Hence there exists constant $m_3(\gamma)>0$ only dependent on γ

such that $|R(z, \hat{z}, \xi, \tau)| \leq m_3(\gamma)|z - \hat{z}||\xi||\tau|$.
$(a6)$ If $a \leq \gamma|\hat{z}| \leq b$ and $\gamma|z| > b$ then

$$R(z, \hat{z}, \xi, \tau) = \big[\phi(z, \xi) - \phi(\hat{z}, \xi)\big]\tau + \Bigg\{6\gamma^5 t_1^z t_2^z \frac{\langle z, \xi\rangle (zz^T)}{|z|^3} + \frac{\gamma}{2} t_1^z t_2^z \big[4\gamma^3|z|\big(\gamma|z| - 1\big) - \gamma^2 t_1^z t_2^z + 1\big]\phi(z, \xi) + \left[\gamma^3|z|^2 - \gamma^2|z| - \frac{1}{2} - \frac{1}{4\gamma}\right]\phi(z, \xi)\Bigg\}\tau.$$

We proceed as in case $(a4)$ and get $\big|\phi(z, \xi) - \phi(\hat{z}, \xi)\big||\tau| \leq m_4(\gamma)|z - \hat{z}||\xi||\tau|$ for some constant $m_4(\gamma) > 0$. For the remaining terms, from $\gamma|\hat{z}| \geq b \geq \gamma|z| \geq a$ it follows $0 \leq |t_2^z| = |\gamma|z| - b| = b - \gamma|z| \leq \gamma|\hat{z}| - \gamma|z| \leq \gamma|\hat{z} - z|$. Besides, $\gamma^3|z|^2 - \gamma^2|z| - \frac{1}{2} - \frac{1}{4\gamma} = \gamma\big[\gamma|z| + \frac{1}{2\gamma}\big]\big[\gamma|z| - 1 - \frac{1}{2\gamma}\big] = \gamma\big[\gamma|z| + \frac{1}{2\gamma}\big]t_2^z$. We process similarly in case $(a5)$ and have

$$|R(z, \hat{z}, \xi, \tau)| \leq m_4(\gamma)|z - \hat{z}||\xi||\tau| + m_5(\gamma)|t_2^z||\xi||\tau| \leq m_6(\gamma)|z - \hat{z}||\xi||\tau|$$

where $m_4(\gamma), m_5(\gamma), m_6(\gamma)$ are positive constants only dependent on γ.
All other cases can be deduced from the previous ones, by an exchanging the roles of z and $\hat{z}$. It is easy to see that the above result also holds in case $z, \hat{z}, \xi, \tau \in \mathbb{R}^N \times \mathbb{R}^N$ ($N \in \mathbb{N}^*$).

$b)$ For $u, \hat{u}, w, v \in \mathbb{L}^2(\Omega)$, $h'_\gamma(u)[w] \in \mathbb{L}^2(\Omega)$ and $\big([h'_\gamma(u) - h'_\gamma(\hat{u})]w, v\big)_{L^2} \leq C_\gamma\|u - \hat{u}\|\|w\|\|v\|$.
$(b1)$ We can write $h'_\gamma(u)[w]$ in the following form

$$\begin{aligned} h'_\gamma(u)[w] &= \left\{\frac{\chi_{\mathcal{A}_u}}{|u|} + \chi_{\mathcal{S}_u}\left[\frac{\gamma}{2} + \left(\frac{2\gamma - 1}{4\gamma} - \frac{\gamma t_1(u)t_2(u) - \gamma^3(t_1(u)t_2(u))^2}{2}\right)\frac{1}{|u|}\right] + \chi_{\mathcal{I}_u}\gamma\right\}w \\ &\quad - \frac{\chi_{\mathcal{A}_u}}{|u|}\chi_{\mathcal{A}_u}\frac{\langle u, w\rangle}{|u|^3}u + \chi_{\mathcal{S}_u}\Bigg[\gamma^2\big(\gamma|z| - 1\big)\big(2\gamma^2 t_1(u)t_2(u) - 1\big) \\ &\quad - \left(\frac{2\gamma - 1}{4\gamma} - \frac{\gamma t_1(u)t_2(u)}{2} + \frac{\gamma^3(t_1(u)t_2(u))^2}{2}\right)\frac{1}{|u|}\Bigg]\chi_{\mathcal{S}_u}\frac{\langle u, w\rangle}{|u|^2}u \\ &=: G_1(u)w + G_2(u)\chi_{\mathcal{A}_u}\frac{\langle u, w\rangle}{|u|^2}u + G_3(u)\chi_{\mathcal{S}_u}\frac{\langle u, w\rangle}{|u|^2}u \end{aligned}$$

where $t_1(u) = \frac{\gamma}{2}\big(\gamma|u| - 1 + \frac{1}{2\gamma}\big)$, $t_2(u) = \gamma|u| - 1 - \frac{3}{2\gamma}$ and $\chi_{\mathcal{A}_u}, \chi_{\mathcal{S}_u}, \chi_{\mathcal{I}_u}$ are defined similarly to $\chi_{\mathcal{A}_z}, \chi_{\mathcal{S}_z}, \chi_{\mathcal{I}_z}$, but for u at each point of Ω. It is easy to verify that $G_1(u), G_2(u), G_3(u) \in L^\infty(\Omega)$. Moreover, we have $\big|\chi_{\mathcal{A}_u}\frac{\langle u,w\rangle}{|u|^2}u|^2, \big|\chi_{\mathcal{S}_u}\frac{\langle u,w\rangle}{|u|^2}u|^2 \leq M^2|w|^2$ for a constant $M > 0$. Hence, $\|h'_\gamma(u)[w]\|_{\mathbb{L}^2} \leq \|G_1(u)\|_{L^\infty}\|w\|_{\mathbb{L}^2} + M\big(\|G_2(u)\|_{L^\infty} + G_3(u)\|_{L^\infty}\big)\|w\|_{\mathbb{L}^2}$ and therefore $h'_\gamma(u)[w] \in \mathbb{L}^2(\Omega)$.
$(b2)$ From the Lipschitz continuity of $h^{"}_\gamma$ (from $\mathbb{R}^2$ to $\mathbb{R}^2$), it follows

$$\big|[h'_\gamma(u) - h'_\gamma(\hat{u})]w\big| \leq \tilde{C}_\gamma|u - \hat{u}||w| \quad \text{a. e. on} \quad \Omega, \quad \tilde{C}_\gamma > 0.$$

Since $[h'_\gamma(u) - h'_\gamma(\hat{u})]w \in \mathbb{L}^2$, we have $\|[h'_\gamma(u) - h'_\gamma(\hat{u})]w\|_{\mathbb{L}^2} \leq \bar{C}_\gamma\|u - \hat{u}\|_{\mathbb{L}^2}\|w\|_{\mathbb{L}^2}$, and also for $v \in \mathbb{L}^2$, we get

$$\big([h'_\gamma(u) - h'_\gamma(\hat{u})]w, v\big)_{L^2} \leq C_\gamma\|u - \hat{u}\|_{\mathbb{L}^2}\|w\|_{\mathbb{L}^2}\|v\|_{\mathbb{L}^2},$$

where $\tilde{C}_\gamma, \bar{C}_\gamma, C_\gamma$ are positive constants only dependent in γ.

$c)$ Use the formula of $h''_\gamma(z)$ and process similarly as in part b), we have that $h''_\gamma(u) \in \mathbb{L}^\infty(\Omega)$. By the Lipschitz continuity of h''_γ (from $\mathbb{R}^2$ to $\mathbb{R}^2$) it then follows that

$$\|h''_\gamma(u) - h''_\gamma(\hat{u})\|_{\mathbb{L}^\infty(\Omega)} \leq M_\gamma\|u - \hat{u}\|_{\mathbb{L}^2}$$

for some constant $M_\gamma > 0$ only dependent in γ. □

References

[1] G. Aubert and L. Vese. A variational method in image recovery. *SIAM J. Numer. Anal.*, 34:1948–1979, 1997.

[2] Kristian Bredies, Yiqiu Dong, and Michael Hintermüller. Spatially dependent regularization parameter selection in total generalized variation models for image restoration. *International Journal of Computer Mathematics*, 90(1):109–123, 2013.

[3] A. Chambolle. An algorithm for total variational minimization and application. *J. Math. Imaging Vision*, 20:89–97, 2004.

[4] A. Chambolle and P.-L. Lions. Image recovery via total variational minimization and related problems. *Numer. Math.*, 76:167–188, 1997.

[5] A. Chambolle, Caselleles V., Cremers D., Novaga M., and T. Pock. *An Introduction to Total Variational for Image Analysis*. De Gruyter Verlag, 2010.

[6] Juan Carlos De los Reyes and Vili Dhamo. Error estimates for optimal control problems of a class of quasilinear equations arising in variable viscosity fluid flow. *Numerische Mathematik*, 2015.

[7] Juan Carlos De los Reyes and Carola-Bibiane Schonlieb. Image denoising: learning the noise model via nonsmooth PDE-constrained optimization. *Inverse Problems and Imaging*, 7(4):1183 – 1214, 2013.

[8] Yiqiu Dong, Michael Hintermüller, and M Monserrat Rincon-Camacho. Automated regularization parameter selection in multi-scale total variation models for image restoration. *Journal of Mathematical Imaging and Vision*, 40(1):82–104, 2011.

[9] Massimo Fornasier, Andreas Langer, and Carola-Bibiane Schönlieb. A convergent overlapping domain decomposition method for total variation minimization. *Numerische Mathematik*, 116(4):645–685, 2010.

[10] K. Frick, P. Marnizt, and A. Munk. Shape constrained regularization by statistical multiresolution for inverse problems. *Inverse Problems*, 28(6):065006, 2012.

[11] K. Frick, P. Marnizt, and A. Munk. Statistical multiresolution dantzig estimation in imaging: Fundamental concepts and algorithmic framework. *Electron. J. Stat.*, 6:231–268, 2012.

[12] K. Frick, P. Marnizt, and A. Munk. Statistical multiresolution estimation for variational imaging: With an application in poisson-biophotonics. *Journal of Mathematical Imaging and Vision*, 46:370–387, 2013.

[13] M. J. Gander. Optimized schwarz methods. *SIAM J. Numer. Anal*, 44(2):1699–1731, 2006.

[14] G. Gilboa, N. Sochen, and Y. Y. Zeevi. Estimation of optimal PDE-based denoising in the snr sense. *IEEE Transactions on Image Processing*, 15:2269–2280, 2006.

[15] K. Gröger. A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. *Math. Ann.*, 283(4):679–687, 1989.

[16] K. Ito and Karl Kunisch. *Lagrange multiplier approach to variational problems and applications*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[17] M. Hintermũller K. Ito and K. Kunisch. The primal dual active set strategy as a semi-smoth newton method. *SIAM Journal on Optimization*, 13:865–888, 2003.

[18] Karl Kunisch and Thomas Pock. A bilevel optimization approach for parameter learning in variational models. *SIAM Journal on Imaging Sciences*, 6(2):938–983, 2013.

[19] Pascal Thériault Lauzier, Jie Tang, and Guang-Hong Chen. Non-uniform noise spatial distribution in ct myocardial perfusion and a potential solution: statistical image reconstruction. In *SPIE Medical Imaging*, pages 831338–831338. International Society for Optics and Photonics, 2012.

[20] Pascal Thériault Lauzier, Jie Tang, Michael A Speidel, and Guang-Hong Chen. Noise spatial nonuniformity and the impact of statistical image reconstruction in ct myocardial perfusion imaging. *Medical physics*, 39(7):4079–4092, 2012.

[21] J. C. De los Reyes and Karl Kunisch. On some nonlinear optimal control problems with vector-valued affine control constraints. in: Optimal control of coupled systems of pde. *International Series on Numerical Mathematics*, 158:105–122, 2009.

[22] J. C. De los Reyes and Karl Kunisch. Optimal control of partial differential equations with affine control constraints. *Control and Cybernetics*, 38:1217–1250, 2009.

[23] E. A. Muravleva and M. A. Olshanskii. Two finite-difference schemes for calculation of bingham fluid flows in a cavity. *Russ. J. Numer. Anal. Math. Model.*, 23(6):615–634, 2008.

[24] Eric Okyere. *Optimized Schwarz Methods for Elliptic Optimal Control Problems*. Erlangung des akademischen Grades, 2009.

[25] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Diferential Equations*. Numerical Mathematics and Scientific Computation, Oxford Science Publications, Oxford, second edition, 1999.

[26] D. Strong, J.−F. Aujol, and T. Chan. Scale recognition, regularization parameter selection, and Meyers G−norm in total variation regularization. *SIAM Journal on Multiscale Modeling and Simulation*, 5:273–303, 2006.

[27] Eitan Tadmor, Suzanne Nezzar, and Luminita Vese. A multiscale image representation using hierarchical (bv, l 2) decompositions. *Multiscale Modeling & Simulation*, 2(4):554–579, 2004.

[28] G. M. Troianiello. *Elliptic differential equations and obstacle problems*. The University Series in Mathematics. Plenum Press, New York, 1987.

[29] C. R. Vogel. *Computational Methods for Inverse Problems.* SIAM, vol. 10, 2002.

[30] Wang Z., Bovik A., Sheikh H.R, and Simoncelli. E.P. Image quality assessment: From error visibility to structural similarity. *IEEE Transactions on Image Processing*, 13:600–612, 2004.

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